Simple Allocation Rules and Optimal Portfolio Choice Over the Lifecycle*

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May 16, 2024

Abstract

In many areas of economics, relatively simple models developed for insight are used as quantitative guides. We study the accuracy of such simple quantitative guidance in an area where it has been widely adopted — lifecycle portfolio choice among stocks, bonds, and liquid accounts — by developing a machine-learning algorithm to solve for optimal portfolio choice in a calibrated lifecycle model that includes many features of reality modelled only separately in previous work. Both for optimizing households and for households that under-save, the average fully-optimal portfolio at each age conforms well to current simple age-dependent prescriptive rules until shortly before retirement, validating existing analyses. We further show that the consumption-equivalent losses from conditioning portfolio shares on age alone are substantial, around 2 to 3 percent of consumption. Fully-optimal equity shares have substantial heterogeneity, particularly by wealth level, state of the business cycle, and dividend-price ratio, implying substantial gains to further customization in these dimensions.

JEL codes: G11, G51, C61, D15

*We thank our discussants James Choi, David Laibson, Peter Maxted, and Alex Michaelides, as well as Hui Chen, Matthias Doepke, Eric French, Francisco Gomes, Fatih Guvenen, Rory McGee, Maarten Meeuwis, Mariacristina De Nardi, Lawrence Schmidt, Chester Spatt, Michele Tertilt, Motohiro Yogo and Yao Zeng for useful discussions on model components. We also thank participants at the 2021 NBER Summer Institute Big Data and High-Performance Computing for Financial Economics Conference, the 2021 European Winter Meetings of the Econometric Society, the 2022 NBER Summer Institute Household Finance Meeting, the April 2023 Micro and Macro Implications of Household Behavior and Financial Decision-Making Virtual Seminar Series, the 2023 NBER Summer Institute Asset Pricing Meeting, and a seminar at Vanguard. Duarte: Gies College of Business, 1206 South 6th Street, Champaign IL 61820, vduarte@illinois.edu; Fonseca: Gies College of Business, 1206 South 6th Street, Champaign IL 61820, juliaf@illinois.edu; Goodman: Investment Strategy Group, The Vanguard Group, Inc. PO Box 2600, Valley Forge PA 19482aaron$_g$oodman2@vanguard.com; Parker: Sloan School of Management, MIT, 100 Main Street, Cambridge, MA 02142, JAParker@mit.edu.

Economic research typically uses parsimonious models to gain insight into observed behaviors or counterfactual outcomes. Even quantitative models are typically parsimonious, designed to deliver understandable rules, and therefore they provide relatively simple guidance. However, in some areas, these relatively simple rules have been adopted by households, firms, or policymakers and used to guide actual behavior, which then raises the important question of whether, in such areas, these simple rules are close to optimal in actual real-world environments. Our starting point for this paper is that big data and recent technological advances in machine-learning tools have led to a leap forward in the complexity of economic problems that can be calibrated and solved. These advances therefore provide economists with another tool with which to evaluate simple rules in more realistic quantitative economic environments. In addition, such tools can potentially provide better or more robust simple rules, as well as quantify the gains from using more complex rules.

In this paper, we apply these ideas to evaluate the simple, age-based portfolio rules currently embedded in investment advice, retirement plan regulation, and investment products like Target Date Funds. These rules, derived from partial-equilibrium lifecycle models of saving and portfolio choice as pioneered in Samuelson (1969) and Merton (1969, 1971), have been widely adopted and have had a large impact on the allocation of household wealth between stocks and bonds (see e.g. Parker, Schoar, Cole, and Simester, 2022).

Using a machine learning solution method, we show that these age-based rules approximate average optimal behavior through the first half of life in the context of a relatively realistic lifecycle model that includes a large number of addition features of investors' environments each of which individually has been shown to be quantitatively important for optimal investor behavior. We also find substantial losses from the use of a simple rule relative to the fully optimum rules, and characterize which state variables drive most of the heterogeneity in optimal behavior.

More specifically, we study the lifecycle optimization problem of a household that consists of a husband and wife who consume both housing and non-housing consumption and are each endowed with a gender-specific earnings profile that has stochastic, left-skewed, serially-correlated shocks. The household allocates its financial wealth among a stock index, a bond index, and a money market account, all with returns that are both serially correlated and correlated with labor income. It can hold these assets in liquid accounts, or save in (pre-tax) retirement accounts with limited employer matching and tax penalties for early withdrawal. For housing, the household chooses between renting, owning with a mortgage, or owning outright, and must pay a cost to sell or to refinance. There is a cash-in-advance constraint and a simple tax and benefit system that includes a consumption floor. During retirement, each individual receives a pension that is a function of lifetime labor income, faces mortality risk and stochastic medical expenses, and gets utility from bequests. We calibrate this model using existing large-sample information, such as income processes based on Social Security earnings records. And we develop and solve the model with a machine-learning-based algorithm, as we describe after our substantive findings.

The simple rules we evaluate are the historically common rule of thumb that a constant two-thirds share of the portfolio be invested in stocks, and a rule that depends only on age and mimics the portfolio of existing (low-fee, index) Target Date Funds (TDFs). Since legal changes allowed them as defaults in retirement plans in 2006, retail investors have increasingly relied on TDFs to allocate their investments, so that by 2019, TDFs accounted for two-thirds of the financial wealth of young investors and TDFs and related balanced funds accounted for more than $4 trillion, or roughly 5% of the US mutual fund market.[1] We compare the consumption-equivalent welfare of the fully-optimal portfolio rules to welfare under each of these two simple alternatives.

We have three main substantive results, all applying to the (relatively well-off) households whose lives we model: two-earner couples that accumulate investable wealth throughout life and do not become extremely wealthy either from inter-generational transfers or own business income.

First, our model suggests that on average, the share of financial wealth that a household should hold in stocks is hump-shaped over the working life, peaking around age 45 at 80% and declining to a stable 60% at and during retirement. However, the average optimal share of *retirement* wealth invested in stocks is quite similar to that of the simple rules for portfolio shares during the working life that are embedded in much financial advising and Target Date Funds. Specifically, for retirement wealth, the average optimal share of retirement wealth held in stocks stays between 80 and 85% until age 50. Similarly, a typical TDF maintains a 90% share in stock until age 40 and then decreases this share smoothly to 75% at age 50. After age 50, the patterns diverge and TDFs hold less equity during

[1] See Parker et al. (2022); Parker, Schoar, and Sun (forthcoming); Investment Company Institute (2020); Morningstar (2020), figures 2.2 and 8.20.

retirement.[2]

Second, the average optimal behavior masks substantial variation. Specifically, the 90th percentile of the cross-household distribution of optimal portfolio share in stocks is close to 100% for wealth in retirement accounts at all ages. The 10th percentile is dramatically lower than the average optimal share, and declines across ages from roughly 30% at age 25 to below 20% during retirement. These first two findings together imply that the portfolios delivered by current portfolio rules that condition only on age are substantially sub-optimal for some investors, and increasingly so as investors age, so that the differences are substantial in the period of life when people have accumulated the most retirement wealth. A further implication is that there is substantial room for improving investor well-being by conditioning current advice or mutual fund offerings on more state variables. We find that differences in wealth levels, the state of the business cycle, and dividend price ratios cause the largest differences in optimal portfolios across households at a given age. These findings quantify the set of strengths and weaknesses of TDFs discussed in Campbell (2016) (Section 5.1) as well as echoing the conclusion of Gomes, Michaelides, and Zhang (2021) that TDFs that take advantage of return predictability would deliver better investor outcomes.

Our third set of results quantifies the loss, relative to optimal behavior, of investing a given age-specific share of a household's retirement wealth in stocks, as in current TDFs. We focus on evaluation of the welfare costs with a discount factor of one, so that we weight flow expected utility in each year of life equally rather than taking a start-of-life perspective as we use to generate decision rules which would imply a low cost of bad late-in-life outcomes. In our model, a household that invests its retirement wealth following the portfolio of the typical (index, low-fee) TDF in expectation loses the equivalent of 1.7% of consumption on average at any given age assuming that it re-optimizes all other behaviors. This loss rises to 2.8% of consumption if we do not allow the household to re-optimize their other behaviors.[3] Strikingly, these losses are similar across the distribution of permanent income, including for lower-income (two earner, stable marriage) families.[4] While TDFs closely mimic average optimal age-contingent allocations, these losses are similar to those

[2]The average optimal share in equity declines linearly to about 60% at retirement, after which it is roughly constant. In contrast, equity shares in TDFs typically decline more rapidly to reach 50% at retirement and then continue to decline slowly after retirement to 30-40%.

[3]Both of these losses are smaller – 0.45% and 0.59% respectively – from the perspective of the household at the start of life due to discounting.

[4]These welfare losses are similar to those calculated by Dahlquist, Setty, and Vestman (2018) in a simpler model moving from age-based to completely optimal rules.

of a simple rule that imposes a constant 2/3 equity share across all states and dates due to the conservative investment strategies of TDFs during retirement.

In the final section of the paper, we revisit these three main findings for relatively impatient households that typically wait until they are older to save in retirement accounts and that accumulate less wealth. In this alternative specification, TDFs continue to track average optimal behavior relatively well for retirement wealth. Because of lower wealth accumulation, average optimal equity shares actually rise with age prior to age 45. After age 45, average optimal equity shares of impatient households are similar to those of more patient households. Further, while optimal equity shares respond to the same state variables as when households accumulate more wealth, they respond by more when households accumulate less wealth. These changes imply that the welfare losses of simple TDF rules relative to optimal behavior are somewhat larger when households are impatient, and that there are larger benefits to further customization of portfolio shares across broad asset classes.

We are able to evaluate the simple rule in a model with more than 20 state variables and shocks by using new tools from the field of deep reinforcement learning, the field of machine learning that studies sequential decision-making. Specifically, building on Duarte (2019), we develop a method that uses deep reinforcement learning to solve dynamic stochastic models like this lifecycle model – models with many states and controls, highly non-linear policy functions, and both discrete and continuous actions – using a policy gradient algorithm (Sutton, McAllester, Singh, and Mansour, 2000). In brief, we parameterize the policy functions over the high-dimensional state space using fully connected feedforward neural networks as cells for two recurrent neural networks (one for working life and one for retirement). We use a stochastic gradient descent algorithm to solve for the parameters of the networks that maximize the expected lifetime utility over a large number of simulated sample paths.

The main advantage of our method over traditional numerical dynamic programming (NDP) is the massive increase in speed that makes it tractable to solve previously infeasible problems. The traditional NDP approach would first characterize the solution to the household's problem by a set of optimality conditions and budget constraints, then construct grids on which choices can be characterized by matrices, and then finally use an optimization algorithm and numerical integration to solve for optimal behavior recursively. In contrast, our method maximizes expected utility using simulated sample paths which avoids computationally-slow numerical integration. And we make use of software (JAX) and hardware developed specifically for machine learning applications, which have been shown considerably reduce computational time even when combined with traditional NDP (Duarte, Duarte, Fonseca, and Montecinos, 2020).

Our method is far easier to use and program (and so less prone to error) than traditional numerical dynamic programming methods. For example, there is no need to specify the density and scale of grids over which policy functions can then be defined as matrices. Finally, this method arguably captures how investors, practitioners, or data scientists actually determine optimal behavior. Optimal behavior is determined by learning from what works for people living their lives according to different portfolio allocation and consumption rules, that is, from watching and learning from the lives of others.

Our methodological contribution adds to a recent and rapidly-evolving literature that develops methods to solve dynamic models using machine learning tools (Duarte, 2019; Scheidegger and Bilionis, 2019; Fernandez-Villaverde, Hurtado, and Nuno, 2020; Maliar, Maliar, and Winant, 2021; Azinovic, Gaegauf, and Scheidegger, forthcoming). Relative to this literature, the tools we use are designed to handle the combination of a large number of state variables and shocks, highly non-linear policy functions, both discrete and continuous actions, and occasionally binding constraints. We highlight the three main challenges we face in applying these increasingly popular methods to solve such a complicated model and how we address them in Section 3.1. While we apply our method to lifecycle consumption and portfolio choice, we conjecture that our method may readily be applied to evaluate the robustness of simple rules in a wide range of partial-equilibrium dynamic problems faced by managers, employers, small open economies, etc.

Our quantitative model of lifecycle portfolio choice also advances the large literature on portfolio choice (see the surveys Curcuru, Heaton, Lucas, and Moore, 2010; Wachter, 2010) and provides a baseline for future quantitative studies of optimal portfolio choice. While our model omits some features of reality, and while each individual ingredient is not new, the model contains the features of the lifecycle problem that we judge to be both most relevant and most important for lifecycle portfolio choice. Further, our model is calibrated using state-of-the art estimates of the deterministic and stochastic processes facing the investor (e.g. DeNardi, French, and Jones, 2010; Wachter, 2010; Guvenen, Ozkan, and Song, 2014). In the conclusion we discuss missing elements and weaknesses that present avenues for future improvements.

While our focus in this paper is on a particular subset of the population and the evaluation of sub-optimal decision rules, our paper contributes to a long line of quantitative analyses of optimal portfolio behavior in lifecycle models. These analyses can be grouped into two different types: those that focuses on inferring features and parameters of the model from observed behavior, and those that offer prescriptive analysis of how investors should allocate their portfolios. Bodie, Merton, and Samuelson (1991), Gakidis (1998), Campbell and Viceira (2002) (Chapter 6), Gomes and Michaelides (2003), Storesletten, Telmer, and Yaron (2004), Cocco, Gomes, and Maenhout (2005), Davis, Kubler, and Willen (2006), Catherine (forthcoming), and Shen (2021) all have advanced the analysis of lifecycle portfolio choice in the presence income risk that cannot be fully hedged. Gomes, Michaelides, and Polkovnichenko (2006) and Dammon, Spatt, and Zhang (2004) considered the role of taxes and tax deferred retirement accounts, Cocco (2005), Hu (2005), and Yao and Zhang (2005) incorporate housing, and Li and Smetters (2011) and Yogo (2016) focuses on the role of Social Security and of medical expenses during retirement respectively.[5]

We cite the main papers that provide the ingredients for our model – both the mathematical structure and the calibrated parameters – as we describe each element of the model. Similarly, we place our method in the machine learning literature when we describe our method in section section 3.1.

1 The Lifecycle Model

Time is discrete and measures age in years. At each age, the household chooses how much to save and how much to consume of non-housing goods and housing. Households receive labor income and earn investment returns that follow known stochastic processes. Optimal consumption and portfolio choices are a set of decision rules for every age that maximize the household's (mathematically) expected present discounted value of utility flows from consumption given knowledge of the structure and parameters of the problem and the history of realizations of stochastic processes up to that age.

Overview The complexity and the realism of the model come from the budget constraint which has the following features. The income process for each spouse consists of a deterministic lifecycle profile subject to negatively skewed persistent and transitory shocks during working life. During retirement, individuals receive pension income based on

[5] More recently Calvet, Campbell, Gomes, and Sodini (2021) include both housing and returns although perfectly correlated. Also of note, a subsection of Cocco et al. (2005) considers the role of medical expenses while Koijen, Nieuwerburgh, and Yogo (2016) considers allocations across annuity and insurance products rather than stocks and bonds.

lifetime earnings and face medical expense shocks. Households face an approximation of the progressive US income tax and benefit system including a consumption floor during retirement. Households can allocate their non-housing wealth among 3 financial assets representing stocks, long-term bonds, and liquid savings. Each returns process consists of an idiosyncratic shock and a loading on a common autoregressive process based on the dynamics of the dividend-price ratio of the aggregate stock market. Correlation among labor income and asset returns is captured by having both the dividend price ratio and the labor income process depend on a stochastic business cycle state. Households can save into liquid financial accounts or retirement accounts, and the model contains a detailed representation of the legal and institutional structure of retirement saving. The household can rent or own housing, and there are adjustment costs associated with changing the size of an owned home or a mortgage. The household faces constraints on borrowing, on short selling, on consumption from a liquid-wealth-in-advance requirement, and on the loan-to-value ratio of its mortgage when purchasing or refinancing. Finally, the household experiences mortality shocks during retirement, and values leaving a bequest at death. The most notable omission from our model is that we study only a traditional family consisting of a man and a woman with fixed ages of retirement and who remain married until death do them part.

We set out the model structure and budget constraints below, starting with the household's working life and then describing retirement. We conclude with the objective function and statement of the complete problem. The calibration of the model is presented in Section 2.

1.1 Working Life

The household consists of two individuals, a man and a woman, indexed by $i \in \{1, 2\}$, who both work from age $t = T_0 = 25$ until the exogenous retirement age $t = T_R = 65$ and who both survive until age T_R with probability one. Households are also differentiated by the deterministic component of each spouse's labor income which depends on their rank in the initial distribution of permanent income, which we denote by q_0^1 and q_0^2.

All random variables sub-scripted by t are realized before the households makes any time-t decisions.

1.1.1 Common Risks

There are two 'aggregate' risks in our partial-equilibrium framework which create correlations among stochastic processes. A stochastic process representing the business cycle generates correlation between labor income and returns on different assets. Second, a stochastic process representing aggregate effective risk aversion generates correlation among asset returns. These two aggregate processes are correlated.

First, the economy can either be in a recession or expansion, with the state variable $e_t = 1$ indicating a recession and $e_t = 0$ indicating an expansion. The economy's state e_t evolves according to the 2x2 transition matrix P_e.

Second, we capture fluctuations in expected returns from serially-correlated fluctuations in the log of the aggregate dividend yield (dividend price ratio) of the stock market, v_t.

$$v_t = \theta_v^1 + \theta_v^2 \Delta e_t^+ + \theta_v^3 \Delta e_t^- + \theta_v^4 v_{t-1} + \epsilon_{v,t}, \tag{1}$$

$$\epsilon_{v,t} \sim N(0, \sigma_v), \tag{2}$$

where $\Delta e_t^+ = \max\{0, e_t - e_{t-1}\}$ and $\Delta e_t^- = \max\{0, e_{t-1} - e_t\}$. The dividend yield follows an AR(1) process, with an intercept that shifts when recessions start or end.

1.1.2 Labor Income

To model the household's labor income, we follow the work done with Social Security earnings records by Guvenen et al. (2014) (GOS) and Guvenen, Kaplan, Song, and Weidner (2017) (GKSW). Each individual's log labor income, $y_t^i = \ln(Y_t^i)$, depends on two factors: a deterministic age profile $\bar{y}_t^i$, and a stochastic process of shocks around this profile that follows GOS's main parametric model. The deterministic profile is a function of gender i, age, and the initial income quantile for each individual, q_0^i, so that $\bar{y}_t^i = \bar{y}(t; i, q_0^i)$. To ease notation, we suppress the dependence of labor income on q_0^i for the rest of Section 1.1.

In each period, log labor income for individual i is equal to the age profile, plus a persistent shock, x_t^i, and a transitory shock, ϵ_t^i.

$$y_t^i = \bar{y}_t^i + x_t^i + \epsilon_t^i, \tag{3}$$

$$x_t^i = \rho_x x_{t-1}^i + \eta_t^i, \tag{4}$$

$$\epsilon_t^i \sim N(0, \sigma_\epsilon), \tag{5}$$

$$\eta_t^i = \begin{cases} \eta_{1,t}^i \sim N(\mu_{\eta_1,e_t}, \sigma_{\eta_1}) \text{ with prob. } p_1 \\ \eta_{2,t}^i \sim N(\mu_{\eta_2,e_t}, \sigma_{\eta_2}) \text{ with prob. } 1 - p_1. \end{cases} \tag{6}$$

The innovation η_t^i in the AR(1) process for x_t^i follows a mixture distribution, which allows for the non-normalities (in particular, negative skewness and excess kurtosis) that Guvenen, Ozkan, and Song document in earnings growth rates.[6] Cyclicality (in particular, countercyclical negative skewness) enters the income process through the dependence of the mean AR(1) innovations, μ_{1,e_t} and μ_{2,e_t}, on the economy's state e_t.

To model taxes and convert the household's gross income to net income, we use the parsimonious tax function introduced by Heathcote, Storesletten, and Violante (2017) (HSV). The household's tax liability as a function of combined gross income, $Y_t = Y_t^1 + Y_t^2$ is given by

$$L(Y_t) = \min\{Y_t - \lambda Y_t^{1-\kappa}, 0.5Y_t\} \tag{7}$$

where the parameter κ controls the progressivity of the tax schedule and the parameter λ shifts the tax function to determine the economy's average tax burden. Heathcote et al. (2017) show that this simple tax function provides a very good approximation to the US tax system (including federal, state, and payroll taxes), and we use the parameter estimates reported in their paper. We cap the tax rate at 50% which occurs for our parameterization just above \$500,000.

1.1.3 Financial Assets

The household's financial wealth is composed of assets held in liquid accounts, a^L, or a (relatively) illiquid retirement account, a^I. The illiquid retirement accounts capture 401(k) and other tax-deferred retirement saving accounts that receive special tax treatment.

[6] And that Schmidt (2016) shows can rationalize asset pricing patterns in a non-lifecycle model.

Contributions to these tax-deferred accounts reduce the household's current tax burden and accumulate tax-free, but are taxed as income when withdrawn and are subject to a 10% penalty if withdrawn before retirement.[7] Households start life with no assets.

In each account, the household can save and invest in $J = 3$ financial assets: short-term government bills (j=1), long-term corporate bonds, and equities.[8] To incorporate realistic return predictability in a tractable way, we follow Wachter (2010) and make each return correlated with the stock market's aggregate log dividend yield, v_t (equations (1) and (2)). The log of the gross return on asset j at (the beginning of) time t, denoted by $R_{j,t}$, is given by:

$$\log(R_{j,t}) = \theta_j^1 + \theta_j^2 \Delta e_t^+ + \theta_j^3 \Delta e_t^- + \theta_j^4 v_{t-1} + \epsilon_{j,t}, \tag{8}$$

$$\epsilon_{j,t} \sim N(0, \sigma_{r,j}), \tag{9}$$

where again $\Delta e_t^+ = \max\{0, e_t - e_{t-1}\}$ and $\Delta e_t^- = \max\{0, e_{t-1} - e_t\}$. Each asset's return depends on an intercept term, which is allowed to differ at the beginning and end of recessions; the aggregate log dividend yield state variable; and a transitory shock.

We denote the wealth in asset class J in each account type, L and I, after returns are realized at (the start of) age t but before age-t saving or withdrawals by $a_{j,t}^L$ and $a_{j,t}^I$ respectively so that:

$$a_t^L = \sum_{j=1}^{J} a_{j,t}^L \tag{10}$$

$$a_t^I = \sum_{j=1}^{J} a_{j,t}^I \tag{11}$$

Define $s_{j,t}^x$ to be the household's net contribution to its holdings of asset j in account type x during period t (i.e., contributions net of withdrawals, after time t returns and incomes are realized and at the same time as consumption and other decisions are being made),

[7] We model traditional accounts only for now, and not the increasingly-common Roth-type accounts. Contributions to Roth accounts do not reduce current taxes, but withdrawals during retirement are tax-free.

[8] Thus we do not consider diversification within stocks or bonds, an issue that can amplify risk (Fagereng, Gottlieb, and Guiso, 2017).

with $s_t^x = \sum_{j=1}^J s_{j,t}^x$. We can then write the state-evolution equation for financial assets as

$$a_{t+1}^L = \sum_{j=1}^J R_{j,t+1}(a_{j,t}^L + s_{j,t}^L), \tag{12}$$

$$a_{t+1}^I = \sum_{j=1}^J R_{j,t+1}(a_{j,t}^I + \Gamma(s_{j,t}^I)), \tag{13}$$

where $\Gamma(\cdot)$ is a function allowing for matching employer 401(k) contributions, with match rate k and limit l until households reach retirement:

$$\Gamma(x) = \begin{cases} x & \text{if } x \leq 0 \text{ or } t \geq T_R \\ (1 + \frac{k}{2})x & \text{if } 0 < x \leq lY_t \\ x + \frac{k}{2} lY_t & \text{if } x > lY_t. \end{cases} \tag{14}$$

We divide the match rate k by 2 to save on state variables and yet to capture the idea that not all jobs match contributions. Annual contributions to tax-deferred retirement accounts are capped as in US law:

$$s_t^I \leq s_{max,t}^I. \tag{15}$$

The contribution limit $s_{max,t}^I$ is constant during the working life, then becomes negative (and age-dependent) beginning at age 70 in order to reflect the IRS schedule of required minimum distributions.[9]

Finally, households cannot short any asset (other than with a mortgage, as described subsequently), so holdings in each asset class and account type must be nonnegative:

$$a_{j,t}^L \geq 0, \tag{16}$$

$$a_{j,t}^I \geq 0. \tag{17}$$

Now define $B_{a,t}$ to be the net contribution to the household's time t budget constraint

[9]We maintain the assumption that households behave optimally other than the constraints we explicitly introduce, so that the primitives of the problem contain no references to default retirement saving rates or portfolios, features that are relevant for actual behavior and welfare (Beshears, Choi, Laibson, and Madrian, 2008; Choukhmane, 2021).

of all decisions regarding portfolio choice and saving in financial assets. We have:

$$B_{a,t} = -s_t^L - s_t^I - L'(Y_t) \sum_{j=1}^{J} \left(\frac{R_{j,t} - 1}{R_{j,t}} \right) a_{j,t}^L + \xi(s_t^I, t) \tag{18}$$

where $L'(Y_t)$ is the marginal tax rate, the first derivative of the tax function,[10]

$$L'(Y_t) = 1 - \lambda(1-\kappa) Y_t^{-\kappa}. \tag{19}$$

and where ξ is a function capturing the tax benefits of contributions to ($s^I \geq 0$) retirement accounts less the tax and possibly early-withdrawal penalty costs associated with withdrawal from ($s^I < 0$) retirement accounts:

$$\xi(s^I, t) = \begin{cases} L'(Y_t) s^I & \text{if } s^I \geq 0 \\ (L'(Y_t) + 0.1) s^I & \text{if } s^I < 0 \text{ and } t < 60 \\ L'(Y_t) s^I & \text{if } s^I < 0 \text{ and } t \geq 60 \\ 0 & \text{otherwise} \end{cases} \tag{20}$$

Net contributions to asset holdings (i.e., savings) enter negatively into the budget constraint, as do taxes incurred on the realized returns on taxable asset holdings. The benefit of contributing to tax-deferred retirement accounts is the reduction in current taxes, while the cost is the taxes on withdrawals plus a 10% penalty associated with withdrawals from the illiquid account prior to age 60.[11]

Finally, we impose a cash-in-advance constraint that reflects the fact that people hold liquidity for transactions related to consumption. We do this in order to capture the transactions demand for liquidity, and to prevent the household from (unrealistically)

[10] Note that for simplicity, and to match the reality that households effectively face constant marginal tax rates with the piecewise linearity of the US tax schedule, we take the marginal tax rate on withdrawals during working life to be the derivative of the tax function evaluated at the household's current labor income (and do not account for second-order effects where withdrawals change income and thereby change the marginal tax rate). Note also that we simply tax realized returns in taxable accounts at the marginal income tax rate rather than including separate dividends or capital gains rates. Since withdrawals from the illiquid retirement account are likely to be a substantial share of income during retirement, we treat withdrawals during retirement equivalently to other income which accounts for second-order effects during retirement (see equation (41)).

[11] Note that, unrealistically, we allow a tax benefit for contributions to retirement plans from age 65 to 69. In our numerical solution however, the average household has a large negative saving rate into retirement wealth immediately before retirement and is dis-saving significantly in the years immediately following retirement.

financing normal consumption expenditures by making repeated withdrawals from its illiquid retirement savings accounts. In particular, as long as the household has any wealth in its illiquid retirement account, it cannot consume more than a fixed multiple of its liquid wealth held in short-term debt:

$$c_t \leq M a^L_{1,t} \text{ if } a^I_t > 0. \tag{21}$$

where c_t denotes non-housing consumption.

1.1.4 Housing

Our model of housing choices is an expanded version of the structure of Berger, Guerrieri, Lorenzoni, and Vavra (2018) (BGLV). Households consume non-housing consumption c_t and housing h_t. At any point in time, the household can choose to either rent or own a home, with $o_t = 0$ indicating renting and $o_t = 1$ indicating ownership. Households start life as renters. The stock of housing owned by the household is given by $\tilde{h}_t$, with $\tilde{h}_t = 0$ when the household is a renter. Thus $h_t = \tilde{h}_t$ if $o_t = 1$ and it will turn out that households choose $h_t > \tilde{h}_t = 0$ if $o_t = 0$. Households who own housing stock may take on mortgage debt, d_t, and they also receive an additional utility benefit from any level h_t of housing.[12] We define by $B_{h,t}(o_{t-1})$ the net contribution to the household's time t budget constraint of all housing decisions which depends on whether the household entered the period as a renter or owner.

If the household is a renter and consumes housing h_t, it pays a rental cost of $\phi p_t h_t$ where ϕ is the constant rent-to-house-price ratio and p_t is the price of housing (relative to the numeraire non-housing consumption good c_t). Following Berger et al. (2018), p_t is a geometric random walk with drift:

$$p_{t+1} = \nu_t p_t, \tag{22}$$

$$\log(\nu_t) \sim N(\mu_\nu, \sigma_\nu). \tag{23}$$

Renters who become homeowners make down payments equal to the cost of the house less the amount of the new mortgage which is subject to a loan-to-value constraint. They must also pay for maintenance on their new house, $\delta \tilde{h}_t$, to cover depreciation.

[12] This utility benefit is not in BGLV and is discussed subsequently as part of the objective function.

Thus, for households that enter the period as renters ($o_{t-1} = 0$), we have:

$$B_{h,t}(0) = -(1 - o_t)\left[\phi p_t h_t\right] - o_t\left[p_t \tilde{h}_t - d_{t+1} + \delta p_t \tilde{h}_t\right] \tag{24}$$

$$d_{t+1} \begin{cases} = 0 & \text{if } o_t = 0 \\ \leq (1-\iota) p_t h_t & \text{if } o_t = 1. \end{cases} \tag{25}$$

The mortgage, d_{t+1}, must stay at zero if the household remains a renter. If the household becomes a homeowner, its mortgage choice is restricted by a loan-to-value constraint: households buying a new house (or refinancing, below) must satisfy $d_{t+1} \leq (1-\iota)p_t h_t$, so that the loan-to-value ratio is at most $1 - \iota$. Note that, in order to match the real-world housing market, the loan-to-value constraint is only imposed when a house is purchased or a mortgage refinanced. An existing homeowner whose house declines in value is never forced to refinance or accelerate payment on its mortgage.

If the household enters the period as an owner and either adjusts its housing stock or becomes a renter, it pays a transaction cost of $f^h p_t h_{t-1}$ (i.e., a fraction f^h of the housing stock it inherited from last period and is now selling, valued at the current house price). Households that remain homeowners must pay maintenance costs. Continuing homeowners must also make mortgage payments if they do not refinance. The amortization schedule of mortgages is described by the parameter χ: a household with current mortgage debt d_t that does not refinance must service its debt by making a payment of at least χd_t. Alternatively, continuing homeowners can adjust their mortgage debt up (refinance) by paying a transaction cost of $f^d p_t h_t$ (a fraction f^d of their house's current value).

Thus, for households entering the period as homeowners, $o_{t-1} = 1$, we have:

$$\begin{aligned} B_{h,t}(1) = -o_t[&\delta p_t \tilde{h}_t + (1 - L'(Y_t))(R_{1,t} - 1 + \Delta r_m) d_t \\ &+ \mathbf{1}_{\{\tilde{h}_t = \tilde{h}_{t-1},\, d_{t+1} > d_t\}} f^d p_t \tilde{h}_t + \mathbf{1}_{\{\tilde{h}_t \neq \tilde{h}_{t-1}\}} p_t(\tilde{h}_t - (1 - f^h)\tilde{h}_{t-1})] \\ &+ (1 - o_t)[(1 - f^h) p_t \tilde{h}_{t-1} - \phi p_t h_t] - (d_t - d_{t+1}) \end{aligned} \tag{26}$$

$$d_{t+1} = \begin{cases} = 0 & \text{if } o_t = 0 \\ \leq (1-\iota) p_t h_t & \text{if } o_t = 1 \text{ and } (\tilde{h}_t \neq \tilde{h}_{t-1} \text{ or } d_{t+1} > \chi d_t) \\ \leq \chi d_t & \text{otherwise.} \end{cases} \tag{27}$$

Depreciation costs enter negatively into the budget constraint, as does tax-deductible mortgage interest (paid at the short-term interest rate plus a mortgage spread of Δr_m)

and principal repayment. If the household adjusts its owned housing stock $\tilde{h}_t$ (either by choosing a new nonzero value or choosing $\tilde{h}_t = 0$ and becoming a renter) it pays the price of its new house (if any) and receives the selling price of its old house net of transaction costs. Finally, if it maintains its current housing stock but refinances its mortgage, it pays refinancing costs and gains the additional funds borrowed in liquid wealth (since the last term, $d_t - d_{t+1}$, is negative). Equation (27) states that if the household becomes a renter then next period's mortgage must be zero. If the household adjusts its housing stock or refinances, then the household can choose any mortgage amount allowable under the loan-to-value constraint. Otherwise, the household can choose any mortgage amount below that required by the amortization rule.

1.1.5 Working-life Objective Function and Unified Flow Budget Constraint

To account for the fact that consumption needs change with family size, consumption enters utility in per-effective-householder form, where the effective size of the household is w_t, the square root of the average family size at each age. As described in Section 2, this profile is such that household size declines to 2 at retirement. Total consumption at each age is a Cobb-Douglas aggregate of non-housing consumption c_t and housing consumption h_t with share parameter α, where housing consumption is the service flow from housing, ϕh_t.[13] We assume constant relative risk aversion utility of total consumption in each year, with risk aversion coefficient γ.

Define the discount factor as β and the retirement value function, which returns the maximized expected discounted utility during retirement, as $V_R^*(\cdot)$. Additionally, as a state variable summarizing the household's liquid financial resources, define cash on hand, Q_t, as the liquid wealth available to the household at the beginning of time t, net of taxes and mortgage interest payments:

$$Q_t = a_t^L + Y_t - L(Y_t) - L'(Y_t) \sum_{j=1}^{J} \left(\frac{R_{j,t} - 1}{R_{j,t}} \right) a_{j,t}^L - (1 - L'(Y_t))(R_{1,t} - 1 + \Delta r_m) d_t \quad (28)$$

Policy functions are functions of the complete set of state variables at the beginning of each age t: $\Xi_t = \left\{ e_t, v_t, \{x_t^i\}_{i=1}^2, Y_t, \{\sum_{t'=T_0}^{t} y_{t'}^i\}_{i=1}^2, Q_t, a_{1,t}^L, a_t^I, p_t, o_{t-1}, h_{t-1}, d_t \right\}$. The sum of each household member's income up to time t is a state variable because several stochastic processes during retirement, discussed below in Section 1.2, depend on individuals' lifetime

[13] ϕ plays no role whatsoever so we set it to the rental rate.

labor earnings. The cash-in-advance constraint in (21) makes the liquid holdings of short-term debt (in addition to cash on hand) a state variable.

The household chooses age-specific policy functions for consumption, portfolio allocations, home ownership, and mortgage debt, $\left\{c_t, o_t, h_t, \{a^L_{j,t+1}\}^J_{j=1}, \{a^I_{j,t+1}\}^J_{j=1}, d_{t+1}\right\}$ as a function of the state variables Ξ_t at each age, which we denote as $\pi_t(\Xi_t)$, to maximize the expected discounted sum of time-separable flow utility. Given the BGLV framework for housing, the consumption aggregate is

$$C_t = c_t^{\alpha}(h_t)^{1-\alpha}, \tag{29}$$

and the household chooses age-specific policy functions π_t to maximize:

$$V^{\pi}(\Xi_{T_0}) = \mathbb{E}\left[\sum_{t=T_0}^{T_R-1} \beta^{t-T_0} \frac{w_t^{\gamma-1} C_t^{1-\gamma}}{1-\gamma} + \beta^{T_R-T_0} V_R^*(\Xi_{T_R}) \;\middle|\Xi_{T_0}\right] \tag{30}$$

subject to the comprehensive flow budget constraint

$$Y_t - L(Y_t) - c_t + B_{a,t} + B_{h,t}(o_{t-1}) = 0, \tag{31}$$

the constraints (16)-(15) and (21), and where the income process is defined in (3)-(6), the tax function is defined in (7), $B_{a,t}$ defined in (18)-(20), and $B_{h,t}(o_{t-1})$ is defined in (24)-(27). We now turn to the retirement value function, $V_R^*(\Xi_{T_R})$.

1.2 Retirement

When the household reaches retirement, it continues to makes consumption, saving, and portfolio-choice decisions to maximize expected discounted utility over the remainder of its life. However, following DeNardi et al. (2010) (DFJ), we make several changes to reflect the changing financial risks that households confront as they age: income becomes deterministic pension payouts rather than stochastic labor earnings, stochastic health costs become a significant household expenditure, and individuals face mortality risk. Let ν_t^i equal 1 if individual i is still alive at time t and 0 otherwise (the exact mortality process for ν_t^i is described below).

1.2.1 Pension Income, Medical Expenditures, and Mortality

Instead of labor income during retirement, each individual receives non-stochastic income representing Social Security and other pension income and faces stochastic expenditure shocks representing medical expenses. Following DFJ, each individual's pension, health, medical expenditure, and mortality processes depend on the individual's permanent income rank. We denote this rank by q_R^i and calculate it as the quantile into which the individual's realized lifetime labor earnings, $\sum_{T_0}^{T_R-1} y_t^i$, fall in the distribution of life-time labor earnings for the distribution including all q_0^i (as described in subsection 2.5).

Pension income is given by

$$Y_t^i = \nu_t^i Y(t; q_R^i), \tag{32}$$

so that income Y_t^i is a deterministic function of age during retirement, although stochastic from the perspective of working life since Y_t^i depends on lifetime earnings. We postpone the treatment of taxes until section 1.2.2 because withdrawals from retirement accounts are taxed as regular income. We set pension income Y_t^i to zero when individual i is no longer alive.

Health shocks and stochastic medical expenditures follow DFJ. We introduce new state variables, g_t^i, representing individual i's current health status: $g_t^i = 0$ indicates "healthy," while $g_t^i = 1$ indicates "sick." Health status transitions depend on previous health status, gender, and age, with

$$\Pr(g_t^i = 1) = P_g(g_{t-1}^i, i, t; q_R^i). \tag{33}$$

Each individual's out-of-pocket medical expenditures m_t^i (i.e., net of Medicare and other insurance) depends on their health status g_t^i. Family medical expenses are the sum of the individual expenses but is capped at $\bar{m}$. The process for medical expenses is given by:

$$m_t = \min[m_t^1 + m_t^2, \bar{m}] \tag{34}$$

$$\log(m_t^i) = \nu_t^i m(g_t^i, i, t; q_R^i) + \nu_t^i \sigma(g_t^i, i, t; q_R^i)\psi_t^i, \tag{35}$$

$$\psi_t^i = \zeta_t^i + \epsilon_{\psi,t}^i, \tag{36}$$

$$\zeta_t^i = \rho_\zeta \zeta_{t-1}^i + \epsilon_{\zeta,t}^i, \tag{37}$$

$$\epsilon_{\psi,t}^i \sim N(0, \sigma_\psi), \tag{38}$$

$$\epsilon_{\zeta,t}^i \sim N(0, \sigma_\zeta). \tag{39}$$

Log medical expenditures are the sum of a health, sex, and age specific mean cost $m(g_t^i, i, t)$ and a shock term ψ_t^i, magnified by a health, sex, and age specific volatility term $\sigma(g_t^i, i, t)$. The medical expenditure state variable ζ_t^i follows an AR(1) process. Medical expenditures are zero once an individual dies, $m_t^i = 0$ if $\nu_t^i = 0$.

Turning to mortality, each person's probability of death depends on their age, sex, income rank at retirement, and health status. If individual i is alive, $v_t^i = 1$, else $v_t^i = 0$. Conditional on having survived through age t, each individual i survives, $v_{t+1}^i = v_t^i$, with the health-, sex-, and age-specific probability $P_d(g_t^i, i, t; q_R^i)$. Following DFJ, anyone still alive at age $T_{max} = 102$ dies with probability 1.

To reflect means-tested government programs such as Medicaid that support the elderly, the household is guaranteed a minimum level of consumption when necessary expenditures like realized medical expenses and mortgage payments are sufficiently high. This government transfer rule is described by

$$n_t = \begin{cases} 0 \quad \text{if} \quad a_{t+1}^L + a_{t+1}^I > 0 \quad \text{or} \quad o_t \neq o_{t-1} \quad \text{or} \quad h_{t+1} \neq h_t \quad \text{or} \quad d_{t+1} \neq \chi d_t \\ \max\left\{0, \frac{\underline{c}}{w_t} + m_t + \tau_t - Y_t + L(Y_t) - B_{a,t} - B_{h,t}(o_t)\right\} \quad \text{otherwise,} \end{cases} \tag{40}$$

where n_t is the government transfer and $\underline{c}$ is the consumption floor (adjusted by the effective consumption shifter). The household is not allowed to receive government transfer payments if it carries any financial wealth to the next period ($a_{t+1}^L + a_{t+1}^I > 0$), adjusts its housing consumption ($h_{t+1} \neq h_t$), changes its home ownership status ($o_t \neq o_{t-1}$), or refinances or makes a larger-than-necessary payment on its mortgage ($d_{t+1} \neq \chi d_t$). Otherwise, when the household's budget constraint is tight enough to push consumption below $\underline{c}$, the government transfer makes up the difference.

Finally, we continue to assume that the household pays taxes during the year as during working life, but also makes an additional tax payment or receives a refund based on the actual tax due calculated from a more accurate version of the true non-linear tax system. As during working life, taxes paid on income in a given year are a non-linear function of Y plus the marginal tax rate (based only on Y) applied to non-retirement capital income, any retirement saving withdrawals, and (as a benefit) mortgage interest payments. However, during retirement, withdrawals of retirement saving may be substantial and they also affect the tax rate on Social Security income. Thus we calculate the complete tax bill that results from including withdrawals from illiquid accounts in the nonlinear tax function. The household must then pay as taxes (or receive as a refund) in the following year any difference between the linear approximation that it pays in t and the actual tax owed based on the non-linear calculation.

The (nonlinear) tax on income and withdrawals from retirement accounts in retirement, $\tilde{L}$, is given by:

$$\tilde{L}(Y_t, s_t^I) = \begin{cases} L\left(\max\{0, -s_t^I\}\right) & \text{if } 0.5Y_t + \max\{0, -s_t^I\} < I_{0.5} \\ L\left(0.5Y_t + \max\{0, -s_t^I\}\right) & \text{if } I_{0.5} \leq 0.5Y_t + \max\{0, -s_t^I\} < I_{0.85} \\ L\left(0.85Y_t + \max\{0, -s_t^I\}\right) & \text{if } 0.5Y_t + \max\{0, -s_t^I\} \geq I_{0.85} \end{cases} \tag{41}$$

which reflects the tax treatment of Social Security benefits: either 0%, 50%, or 85% of Social Security income is taxable, depending on the level of "combined income" (half of Social Security income plus illiquid asset withdrawals).[14] The difference between this and what is paid during the year is the tax bill (or refund) that the household must pay in $t+1$:

$$\tau_{t+1} = \tilde{L}(Y_t, s_t^I) - L(Y_t) + \xi(s^I, t). \tag{42}$$

1.2.2 Financial Assets

During retirement the household does not face withdrawal penalties on its illiquid assets nor does it receive employer matching on retirement contributions (as stated for $t \geq T_R$ in Equations 14 and 20). However, it must pay income taxes on withdrawals from

[14]This is only approximately correct because non-asset income, Y_t, from DFJ includes not only Social Security benefits but also defined-benefit pensions and annuities which are actually taxed as ordinary income (in the same way as illiquid asset withdrawals $-s_t^I$ in our model). This approximation is close however because Social Security is all or most of Y for households over 65, averaging 3.5 times the income from private pensions and annuities (Social Security Administration, 2016, pages 222-223).

the retirement account (as given in equation 20) and is subject to the IRS's rules regarding required minimum distributions from retirement accounts after age 70 (equation 15). Thus, $B_{a,t}$, as defined in Equation 18, summarizes the contribution of financial assets to the budget constraint.

1.2.3 Housing

During retirement, the part of the budget constraint relating to housing and mortgages, $B_{h,t}(o_{t-1})$, as well as the constraints on these choices are exactly the same as during working life.

1.2.4 Objective Function and Unified Flow Budget Constraint

We can now write down a complete statement of the household's problem during retirement. Following DFJ, we define utility over bequests, b_t, as

$$u_b(b_t) = \tilde{b}\frac{(b_t + \underline{b})^{1-\gamma_b}}{1-\gamma_b}, \tag{43}$$

which the household receives when the longer-lived spouse dies at age t and where $\tilde{b}$ captures the intensity of the bequest motive, $\underline{b}$ shifts the curvature of utility function and allows bequests to be a luxury good, and γ_b is risk aversion over bequests. Mortality is realized at the beginning of the period, so the bequest amount is given by

$$b_t = a_t^L + a_t^I - L\left(\frac{a_t^I}{5}\right) - \tau_t + \max\{(1-\delta-f^h)p_t\tilde{h}_{t-1} - d_t, 0\}, \tag{44}$$

where $a_t^L + a_t^I - L\left(\frac{a_t^I}{5}\right)$ gives financial assets at the beginning of the period net of taxes paid by the recipient on inherited pre-tax wealth,[15] τ_t gives the carry-forward tax bill determined in the previous period, and the final term gives the current value of the owned housing stock $\tilde{h}_{t-1}$ carried from the previous period (net of the beginning-of-period mortgage debt, depreciation, and the transaction costs associated with selling the house). The maximum operator around the last term means that underwater mortgages (i.e., $(1-\delta-f^h)p_t\tilde{h}_{t-1} - d_t < 0$) cannot cause the household's bequest to be less than its

[15] Under current US tax law, inherited 401(k) or IRA wealth must be withdrawn over 10 years and is taxed as income when withdrawn. Because generally the inheritor will also have labor income, we suppose a tax bill as if the inheritor withdrew over only 5 years.

financial wealth net of taxes. Additionally, if the carry-forward tax bill is large enough to make the bequest value in (44) negative, we set it to zero.

As in the pre-retirement period, consumption needs depend on household size. When both members of the household are still alive ($v_t^1 = v_t^2 = 1$), $w_t = \sqrt{2}$. When the shorter-lived spouse dies, $w_t = 1$, reflecting the one-member household's diminished consumption needs. When the second spouse dies, $w_t = \sqrt{v_t^1 + v_t^2} = 0$. We also allow consumption utility to depend on the health status of household members. In particular, to reflect the difficulty of maintaining a home and the increased likelihood of moving to assisted living environments when in poor health, the utility benefit of homeownership relative to renting decreases if the household's members are in the "sick" state. We implement this utility reduction by multiplying homeowners' housing consumption h_t by a scaling factor $\tilde{u} < 1$, raised to the number of household members who are sick so that we redefine the consumption aggregate as

$$C_t = c_t^{\alpha} (\tilde{u}^{o_t (g_t^1 + g_t^2)} h_t)^{1-\alpha}. \tag{45}$$

This equation collapses to (29) during working because we assume everyone is healthy so that $g_t^1 + g_t^2 = 0$.

The set of state variables during retirement is different from that during working life in two ways. First, in place of the state variables for forecasting labor income, x_t^i, and pension income, $\sum_{t_0}^{t} y_t^i$, are state variables for summarizing pension income and forecasting medical expenses, q_R^i and ζ_t^i. Second, the cash-on-hand state variable must account for medical expenses m_t and the previous year's owed taxes τ_t. Denoting the modified retirement cash-on-hand variable as Q_t^R, this gives

$$Q_t^R = Q_t - m_t - \tau_t, \tag{46}$$

where Q_t is defined in (28). Finally, the health status (g_t^i) and survival (ν_t^i) states of the household's members are now state variables.[16]

Given Ξ_{T_R} at retirement, the household chooses the set of policy functions for consumption, portfolio allocations, home ownership, and mortgage debt as a function of Ξ_t, which

[16]Thus, during retirement the state at age t is given by $\Xi_t = \left\{ e_t, v_t, Y_t, Q_t^R, a_{1,t}^L, a_t^I, \{g_t^i\}_{i=1}^2, \{q_R^i\}_{i=1}^2, \{\zeta_t^i\}_{i=1}^2, \{\nu_t^i\}_{i=1}^2, p_t, o_{t-1}, h_{t-1}, d_t \right\}$.

we again denote by $\pi_t(\Xi_t)$ at every age to maximize:

$$V_R^{\pi}(\Xi_{T_R}) = \mathbb{E}\left[\sum_{t=T_R}^{T_{max}} \beta^{t-T_0}\left(\mathbf{1}_{\{w_t>0\}}\frac{w_t^{\gamma}C_t^{1-\gamma}}{1-\gamma} + \mathbf{1}_{\{w_t=0,\, w_{t-1}>0\}}\tilde{b}\frac{(b_t+\underline{b})^{1-\gamma}}{1-\gamma}\right) \;\middle|\; \Xi_{T_R}\right] \tag{47}$$

subject to a modified version of the budget constraint during the working life (31) which accounts for the fact that the household faces medical expenditures, may receive government transfers, and pays any additional tax liabilities from the previous year:

$$Y_t - L(Y_t) - c_t - m_t + n_t - \tau_t + B_{a,t} + B_{h,t}(o_{t-1}) = 0. \tag{48}$$

The household is subject to the contribution and withdrawal constraints (16)-(15) and (21). Income is given by (32), the tax function is defined in (41), the process for medical expenditures is defined in (34)-(39), the benefit n_t is defined by (40), b is given by (44), $B_{a,t}$ and $B_{h,t}(o_{t-1})$ are defined as during working life by (18)-(20) and (24)-(27), and finally consumption is given by (45).

The retirement value function referenced in the working-life objective (30), $V_R^*(\Xi_{T_R})$, is given by the maximized value of (47).

2 Model Parameters

The complexity of our setup creates a large number of parameters that we must calibrate or estimate before proceeding to solve the model. Wherever possible, we take parameter values directly from the same literature that we use to construct the model itself.[17] We discuss our parameterization decisions here and provide a complete list of parameter values in Table I. All dollar amounts in the paper are given in 2013 dollars and are inflation-adjusted using the PCE deflator (following GKSW).

We parameterize the model to match a married couple both born in 1924 (the youngest cohort included in DFJ's data) and their working lives run from age 25 (in 1949) to age 65 (in 1989). We adjust all dollar amounts to 2013 dollars. Households begin life as renters with no financial assets, no housing wealth, no mortgage debt, and the persistent component of labor income variance $x^i_{T_0}$ set to 0.

[17]The online appendix contains the inputs that cannot be neatly listed in Table I because they are vectors or matrices.

2.1 Common Risks

We use official NBER recession-dating betweeen 1915 and 2015 to estimate the recession transition matrix P_e. To estimate the dividend yield process in (1) we use data on U.S. dividend yields during the 1915-2015 period from Jorda, Kroll, Kuvshinov, Schularick, and Taylor (2019).

2.2 Labor Income and Taxes

We take all parameter values for the labor income process in (3)-(6) directly from GOS.

To specify the deterministic age profile, we use the summary statistics that GKSW report in their appendix. In particular, GKSW compute a series of quantiles (the 25th, 50th, 75th, and 90th) of the log income distribution, by gender, age (between 25 and 55) and cohort (those that turned 25 between 1957 and 1983). After specifying a gender i, an initial income quantile q_0^i, and a cohort k, we simply read off the relevant 31-observation time series from GKSW and use it as our age profile, so that $\bar{y}_t^i = \bar{y}(t; i, q_0^i, k)$.[18]

Because the oldest cohort for which GKSW report income summary statistics turned 25 in 1957, and because GKSW only report summary statistics through age 55, we must perform some interpolation to populate our deterministic income profile for the 25-32 and 56-65 age ranges. For the 25-32 age range, we simply use the data that GKSW report for the 1957 cohort, adjusted by the average annual rate of real wage growth during the 1950s.[19] For the 56-65 age range, we use the predicted values obtained from regressing the GKSW income data between ages 25 and 55 on a second-order polynomial of age.

The tax progressivity parameter κ comes directly from HSV, and we set the shift parameter λ so that tax liability becomes positive at the same real income level as in HSV.

[18] In their baseline sample, GKSW impose minimum-participation requirements and include only individuals that earn sufficiently high income in a sufficient number of years. We use the male income profile derived from GKSW's baseline sample. However, given that our targeted cohorts worked during a period of rising female labor force participation, we prefer not to assume that all of our simulated households have a second earner working full-time during the entire working life. We therefore use the female income profile derived from an expanded sample in GKSW that does not impose minimum-participation requirements. This allows us to capture empirical patterns of female labor-force participation during the period of study.

[19] We obtain real wage growth rates from Peake and Vandenbroucke (2020).

Table I: Parameter Values

Parameter	Value	Description	Source
T_0	25	beginning of working life	calibrated
T_R	65	retirement age	calibrated
T_{max}	102	maximum attainable age	DFJ
P_e	see appendix	business cycle transition matrix	estimated
$\theta_v^1 - \theta_v^4$	see appendix	dividend yield process	estimated
$\theta_j^1 - \theta_j^4$	see appendix	return processes for $j \in \{\text{long bond, short bond, equity}\}$	estimated
ρ_x	0.979	labor income persistence	GOS
$\mu_{\eta_1,0}$	0.119	first labor income innovation mean, expansion	GOS
$\mu_{\eta_1,1}$	-0.102	first labor income innovation mean, recession	GOS
$\mu_{\eta_2,0}$	-0.026	second labor income innovation mean, expansion	GOS
$\mu_{\eta_2,1}$	0.094	second labor income innovation mean, recession	GOS
σ_{η_1}	0.325	first labor income innovation std. dev.	GOS
σ_{η_2}	0.001	second labor income innovation std. dev.	GOS
σ_ϵ	0.186	labor income transitory shock std. dev.	GOS
p_1	0.490	labor income mixture probability	GOS
λ	5.5	income tax level	HSV
κ	0.181	income tax progressivity	HSV
k	0.5	employer 401(k) match rate	calibrated
l	0.06	employer 401(k) match limit	calibrated
$s^I_{max,t}$	see appendix	401(k) contribution limit & req. min. distributions	US tax code
M	12	cash-in-advance constraint on liquid wealth	calibrated
δ	0.022	housing stock depreciation rate	BGLV
ϕ	0.060	rent-to-house-price ratio	BGLV
f^h	0.050	housing stock adjustment transaction cost	BGLV
f^d	0.012	mortgage refinancing transaction cost	BGLV
χ	0.969	mortgage amortization speed	BGLV
ι	0.100	mortgage down payment requirement	BGLV
μ_ν	0.012	mean house price innovation	estimated
σ_ν	0.039	house price innovation std. dev.	estimated
Δr_m	0.03	mortgage interest rate spread	calibrated
ρ_ζ	0.922	medical expenditure shock persistence	DFJ
σ_ζ	0.224	medical expenditure persistent shock std. dev.	DFJ
σ_ψ	0.815	medical expenditure transitory shock std. dev.	DFJ
$\bar{m}$	1,000,000	family medical expenditure cap	calibrated
$\underline{c}$	\$3,957	retirement consumption floor	DFJ
$\tilde{b}$	23.6	bequest utility intensity	DFJ (2016)
$\underline{b}$	369,000	bequest utility intercept	DFJ
$\tilde{u}$	0.90	homeownership scaling factor for poor health	calibrated
$I^m_{0.5}$	\$32,000	Social Security 50% tax cutoff for married households	US tax code
$I^m_{0.85}$	\$44,000	Social Security 85% tax cutoff for married households	US tax code
$I^s_{0.5}$	\$25,000	Social Security 50% tax cutoff for singles	US tax code
$I^s_{0.85}$	\$34,000	Social Security 85% tax cutoff for singles	US tax code
α	0.888	Cobb-Douglas utility share of non-housing consumption	BGLV
γ	3.84	relative risk aversion	DFJ
β	0.96	discount factor	calibrated
w_t	see appendix	effective family size scaling factors	calibrated

Notes: The following process from DFJ are presented in the appendix: deterministic retirement pension income profiles, $Y(t; q_R^i)$; health status transition probabilities, $P_g(g_{t-1}^i, i, t; q_R^i)$; mean medical expenditure profiles, $m(g_t^i, i, t; q_R^i)$; medical expenditure volatility profiles, $\sigma(g_t^i, i, t; q_R^i)$; survival probabilities, $\pi(g_t^i, i, t; q_R^i)$. Additional parameter values are as listed in the text of Section 2.

2.3 Financial Assets

We estimate the asset return processes in (8) with U.S. data over the 1915-2015 period. We take data on short-term government bills and equities from Jorda et al. (2019) and use the Dow Jones Total Corporate Bond index to estimate the long-term corporate bond process.

We set $k = 0.5$ and $l = 0.06$ in the employer-matching rule (14) to reflect the common employer policy of matching 50% of contributions up to 6% of income. As reflected in (14), we divide k by 2 on the assumption that only one spouse has access to employer matches at any given time, yielding an effective match rate of 0.25. We set the 401(k) contribution limit $s^I_{max,t}$ for $t < T_R$ in equation (15) to $\$20,603$ reflecting the highest IRS limit that our target cohort experienced during their working lives (in 1989), adjusted for inflation and multiplied by 2 to account for the two-member household.[20] We calibrate the cash-in-advance parameter M to 12 so that the household must carry the equivalent of 1 months' worth of consumption in safe, liquid wealth.

2.4 Housing

We take all parameter values for the housing part of the model directly from BGLV, except for two small changes. First, we estimate the μ_v and σ_v parameters governing the house price process using the housing capital gain series in Jorda et al. (2019) since 1985. Second, we add the mortgage interest spread Δr_m and set it to 0.03.

2.5 Retirement

We take all parameter values for the pension income, health status, medical expenditures, and mortality processes directly from DFJ. The tax liability parameters $I_{0.5}$ and $I_{0.85}$ in (41) and the required minimum distributions ($s^I_{max,t}$ for $t \geq T_R$ in equation (15)) are set based on IRS regulations.

We determine q^i_R as follows. For each initial quantile q^i_0, we simulate paths for income during the working life from the earnings process (described by equations (3)-(6)) and combine to obtain a simulated distribution of cumulative, working-life incomes. At retirement, we compare each individual's lifetime earnings, $\sum_{t=T_0}^{T_R-1} y^i_t$, to this distribution to determine their lifetime earnings quantile, q^i_R.

[20] In reality, the maximum contribution rises each year but we have a real not a nominal model.

We take our process for pension income from DFJ, who estimate the age path of the combined value of "Social Security benefits, defined-benefit pension benefits, annuities, veteran's benefits, welfare, and food stamps." We also use their estimated processes for health shocks and medical costs. We set $Y(t; q_R^i)$ and the medical expense processes based on the permanent income quantile closest to q_R^i for each individual.[21]

2.6 Objective Functions

Following DFJ, we set both coefficients of relative risk aversion (γ for the consumption utility function and γ_b for the bequest utility function) to 3.84. We also use DFJ's estimated values for the bequest utility parameters $\tilde{b}$ and $\underline{b}$ (adjusting the latter for inflation).[22] Finally, we set the discount factor β to 0.96.

To construct our age profile of family size, we follow Salcedo, Schoellman, and Tertilt (2012), who use decennial census data to compute average family sizes for 5-year age buckets (see their Figure 3). We set our family size equal to 2 at age 65, then apply the average growth rates implied by the estimates in Salcedo et al. (2012) to fill in our profile back to age 25. w_t is the square root of this average family size.

We set the homeownership scaling factor $\tilde{u}$ to 0.9, so that effective housing consumption is reduced by 10% if the household is a homeowner and one member is sick, and by 19% if the household is a homeowner and both members are sick.

3 Solution Method

To solve the investor's optimization problem, we build on the framework of Duarte (2019) which develops a method to use policy gradient algorithms to solve high-dimensional problems in economics and finance. We apply and develop machine learning tools, and take advantage of both recent hardware and software development for machine learning. We parameterize the investors' policy functions as fully connected feedforward neural networks and update the networks' parameters with stochastic gradient descent to maximize

[21] While DFJ uses permanent income quintiles, DFJ provided us with their processes at each fifth percentile, and we choose the DFJ quantile in $\{0, .05, .1, ..., .95, 1\}$ that is closest to each q_R^i. These processes are in our supplemental materials.

[22] De Nardi, French, and Jones (2016a) find somewhat different parameter values for the bequest function for risk aversion closer to 3. De Nardi, French, and Jones (2016b) discuss how the parameters of the bequest function are not well identified due to precautionary saving motives, at least when identification comes primarily from the structural model and data on saving choices.

the expected value of the stochastic reward as captured by the investor's value function (see Sutton and Barto, 2018). Relative to a more traditional numeric dynamic programming, we avoid defining policy functions over grids of state variables and performing numerical integration to compute expectations.

In this section, we expand on this solution algorithm and describe the three key challenges we face in applying standard machine learning techniques and how we address them. We then present the architecture of our neural networks, which are chosen due to their computational efficiency, and describe how we implement this solution algorithm.

3.1 Algorithm

We start by describing a standard policy gradient algorithm (Sutton et al., 2000). Following Section 1, $\pi_t(\Xi_t)$ is an age-specific policy function representing the choices of a household as a function of the set of state variables Ξ_t. Our goal is to find a policy function that maximizes expected lifetime utility $V^{\pi}(\Xi_{T_0})$, which we define in equation (30). To do so, we parameterize the set of lifetime policy functions, π, using a neural network

$$\pi^{\Theta}(\Xi) \equiv \pi(\Xi;\Theta). \tag{49}$$

where Θ is the collection of parameters for all networks. For ease of notation, we refer to the collection of all the shocks as ε and denote the sum of *realized* discounted utilities given a collection of parameters and shocks as $R(\Theta,\varepsilon)$. Given this parametrization, we define the loss function as

$$\mathcal{L}(\Theta) = -V^{\pi^{\Theta}}(\Xi_{T_0}) = -\mathbb{E}_{\varepsilon\sim\mathcal{D}}\left[R(\Theta,\varepsilon)\right], \tag{50}$$

where $\mathcal{D}$ denotes the distribution from which shocks and initial states are sampled.

The set of parameters Θ is chosen to minimize the loss function. To find this minimum, we use stochastic gradient descent and proceed iteratively. For a set of initial parameters Θ, assuming that the loss function is differentiable with respect to Θ, one step of standard (non-stochastic) gradient descent adjusts the parameters according to the following equation

$$\Delta\Theta = -\alpha\nabla_{\Theta}\mathcal{L}(\Theta), \tag{51}$$

where α denotes the learning rate and ∇_{Θ} denotes the gradient with respect to Θ. The idea behind gradient descent is to adjust network parameters Θ in the direction that (locally) reduces the loss function the fastest.[23] Moreover, because it would be very computationally costly to compute the above expectation, stochastic gradient descent algorithms approximate this expectation with a small i.i.d. sample of simulated paths.

There are three challenges that arise in applying a standard policy gradient algorithm to solve our complex dynamic stochastic problem (and potentially many other problems in economics and finance). First, since some of the choices represented by π are discrete, the loss function given by Equation (50) is not differentiable with respect to Θ. The second challenge, which broadly affects Deep Reinforcement Learning, is that this algorithm might prematurely converge to a local optimum if there is not sufficient exploration. Third, approximating gradients using simulated data introduces variance in these estimates that can lead to slow convergence, especially in environments such as ours with a large finite horizon and high-variance, persistent shocks.

We circumvent the first challenge by using Parameter-Exploring Policy Gradient methods (Sehnke, Osendorfer, Rückstieß, Graves, Peters, and Schmidhuber, 2010). Instead of searching for a single vector of parameters Θ, these methods consist of searching for a distribution of parameters. Specifically, we assume

$$\Theta \sim N(\mu, \sigma^2 I), \tag{52}$$

where μ is a vector with the same dimensions as Θ, σ is a scalar, and I is the identity matrix. We chose μ so as to minimize the loss function, which we can now define as

$$\mathcal{L}(\mu) = -\mathbb{E}_{\Theta \sim N(\mu,\sigma), \varepsilon \sim \mathcal{D}} \left[R(\Theta, \varepsilon) \right]. \tag{53}$$

As in Salimans, Ho, Chen, Sidor, and Sutskever (2017), we treat σ as a fixed hyperparameter, initially setting σ to 0.001 and annealing it to zero during training. Importantly, as σ approaches zero, our algorithm settles on a deterministic vector of parameters Θ and policy functions map a set of state variables into deterministic actions. Sehnke et al. (2010) show that the loss function given by Equation (53) is differentiable with respect to μ even when the original loss function given by Equation (50) is not, and is given by

$$\nabla_\mu \mathcal{L}(\mu) = -\mathbb{E}_{\Theta \sim N(\mu,\sigma), \varepsilon \sim \mathcal{D}} \left[\nabla_\mu \log \varphi(\Theta) R(\Theta, \varepsilon) \right], \tag{54}$$

where φ is the probability density function of Θ.

[23] For a more detailed description of this approach, see Duarte (2019).

As discussed, because it is computationally infeasible to solve our model by repeatedly computing the expectation in the equation above, we employ stochastic gradient descent and approximate this expectation with a small i.i.d. sample of simulated paths. We adjust μ at each iteration by

$$\Delta\mu \approx \alpha \frac{1}{N} \sum_{i=1}^{N} \nabla_{\mu} \log \varphi(\Theta_i) R(\Theta_i, \varepsilon_i) \tag{55}$$

and proceed by adjusting parameters according to this equation at each iteration until a suitable convergence criteria is met.

We overcome the second challenge by relying on state-of-the-art techniques to encourage exploration. First, we add a term to the objective function that is proportional to the entropy of the policy function π to reward exploration (Williams and Peng, 1991; Mnih, Badia, Mirza, Graves, Lillicrap, Harley, Silver, and Kavukcuoglu, 2016). Since portfolio shares sum to one, we compute the entropy of a policy function representing portfolio choice at time t as $H(\pi, t) = -\sum_{j=1}^{J} s_{j,t}^{x} \log s_{j,t}^{x}$ for $x \in \{L, I\}$. By adding a term proportional to $H(\pi, t)$ to the objective function, we reward policy functions that imply that positive amounts are invested in all asset classes, which encourages exploration.[24] The parameter governing the strength of this entropy boost is reduced to zero during training, eliminating the incentive to invest in all asset classes as we approach the optimal solution.

Moreover, we employ ϵ-greedy exploration (Mnih, Kavukcuoglu, Silver, Graves, Antonoglou, Wierstra, and Riedmiller, 2013) in all discrete choices. This means that, during training, the investor takes a random action with probability ϵ, ensuring that they explore all actions with positive probability. The ϵ probability is also reduced to zero during training and there is no entropy bonus or probability that the agent takes a random action when we evaluate the model.[25]

Finally, we overcome the third challenge by using a variance-reduction technique known in the Evolution Strategies literature as mirrored sampling.[26] For every Gaussian vector of shocks ε, we always evaluate pairs of perturbations ε and $-\varepsilon$. This technique has been shown to reduce variance while still producing unbiased estimates (e.g. Brockhoff,

[24]As we discuss in Section 3.2, our network architecture is such that consumption, financial investment, housing, and debt repayment are represented as fractions of disposable income that, because the budget constraint must be satisfied, must sum to one. That means that we can employ an analogous procedure for the policy function representing these choices and, consequently, apply an entropy boost to all choice variables.

[25]We discuss how we tune hyperparameters in Section 3.2.

[26]See Salimans et al. (2017) for a recent application

Auger, Hansen, Arnold, and Hohm, 2010). In addition to mirrored sampling, some of the features of our network architecture, which we describe below, also help address the issue of high variance.

Together, these three solution strategies address the three concerns, and make it feasible to solve our lifecycle model – and presumably similar models – that were previously computationally infeasible to solve. These tools work in concert with the choices of network architecture that we now describe.

3.2 Network Architecture

In models like our lifecycle model, the multi-period horizon, the substantial variance shocks, and multi-dimensional state space all increase the variance of the gradients used to determine how to optimally adjust network parameters. While the algorithm just discussed in the previous subsection partly mitigates this potential problem, two features of our network architecture are also deigned with this concern in mind.

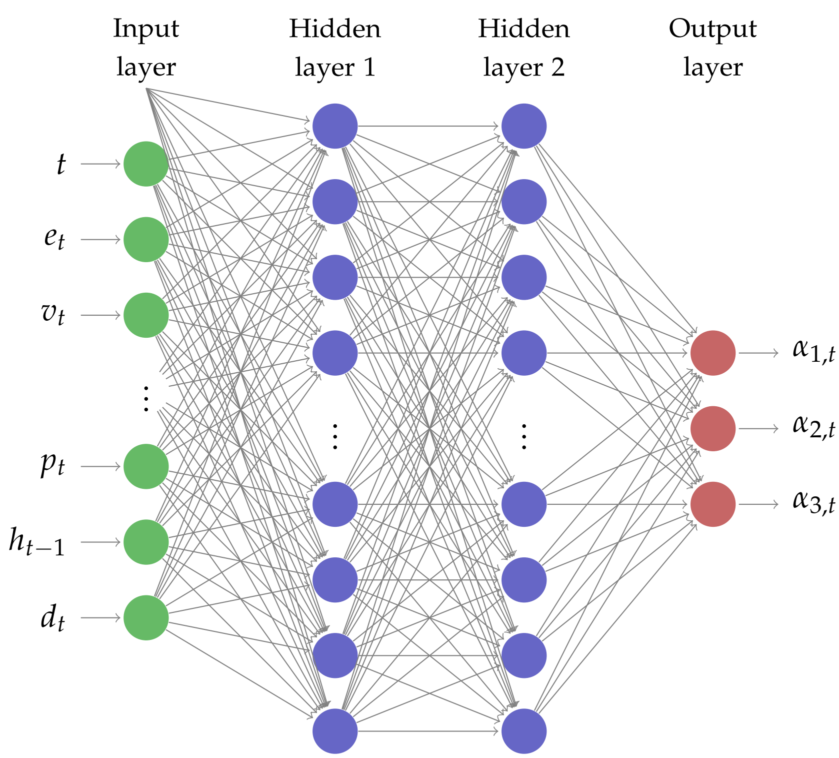
First, we parameterize policy functions using two neural networks, one for the working age period and one for the retirement period. Using two networks instead of one effectively reduces the time horizon faced by the retirement-period network, which helps reduce the variance of gradients. Further, having separate networks allows us to flexibly address the fact that the set of control variables and state variables both change at retirement.

The second aspect of our network architecture that helps reduce the variance of gradients is that, instead of having a separate network for each time t, time is an input of our networks. We choose this structure following research on parameter sharing (Caruana, 1993) which takes advantage of the fact that an output at time t will be very similar to an output at time $t+1$. Thus, treating time like any other state in the network reduces the number of network parameters and increases efficiency (Tsitsiklis and Van Roy, 2001). In particular, this architecture increases the magnitude of gradients without increasing their variance – since changes to a parameter will affect actions across all time periods – and thus leading to a better approximation of gradients with a small i.i.d. sample of simulated paths.

We automatically tune hyperparameters using Population Based Training (Jaderberg, Dalibard, Osindero, Czarnecki, Donahue, Razavi, Vinyals, Green, Dunning, Simonyan, Fernando, and Kavukcuoglu, 2017). This technique trains multiple neural networks in parallel that differ in their hyperparameters and periodically assesses the performance of these networks. At each assessment step, the algorithm interrupts the training of poorly performing networks and replaces them with perturbed copies of high-performing networks, automatically directing computational resources to promising sets of hyperparameters.

Figure I shows our network structure for the case of the portfolio choice of the household during working life. As discussed in Section 3.1, this network takes as inputs (represented as green nodes in Figure I) time (t) and the set of state variables. It produces as outputs (represented as red nodes) the portfolio shares of the three assets. This network illustrates two other properties of our network architecture.

Figure I: Network Architecture



Note: This figure illustrates the architecture of a neural network representing the portfolio choice at time t of an agent during working life.

First, every possible action the agent takes is modeled as a sequence of two hidden layers and one output layer. In Figure I)the two hidden layers are represented by blue nodes. Each node in the first hidden layer is a non-linear transformation of a linear combination of the inputs. We normalize inputs using a slow-moving average of inputs over the first

10,000 iterations.[27] Each node in the second hidden layer is a non-linear transformation of a linear combination of the nodes of the first hidden layer. Finally, each output (or node) of the output layer is a non-linear transformation of a linear combination of the nodes of the second hidden layer. These non-linear transformations are commonly referred to as activation functions.

Second, portfolio share must be posotive and sum to one, and we make use of the softmax function to ensure that such constraints hold.[28] The softmax function is a function $\sigma : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that

$$\sigma(x)_i = \frac{e^{x_i}}{\sum_{j=1}^{n} e^{x_j}}, \text{ for } i \in \{1, \ldots, n\} \tag{56}$$

The usefulness of this transformation is immediate in our portfolio choice example in Figure I. The no-borrowing constraints of Equations 16 and 17 imply that portfolio shares must be non-negative and sum to one, which the softmax function imposes. We also use a softmax activation function when representing a household's decision to split its disposable income between consumption, financial investment, housing, and debt repayment. This functional form imposes that each of these variables will be a fraction of disposable income and, consequently, the budget constraint will always be satisfied. The activation function for the output layer associated with discrete choices is also a softmax, and the agent chooses the discrete case associated with the highest output.

3.3 Solution

We run code to implement this solution in Google's TensorFlow Research Cloud, a cloud service intended for researchers.[29] At each iteration, we sample 2,048 vectors of parameters according to the distribution given by Equation 52.[30] As we describe above, a vector of parameters determines the policy functions of an investor and, for each vector of parameters, we simulate the lives of 256 investors who make decisions according to those policy functions. This amounts to simulating the lives of 524,288 investors (256$\times$2,048) per

[27] As in any application using neural networks, it is important for our algorithm that inputs have similar magnitudes. In our setting, some inputs are endogenous, so we do not know ex ante their respective means and variances. We thus use a slow-moving average of simulated states in this normalization.

[28] For the hidden layers, we use a sigmoid linear unit (SiLU) activation function.

[29] The TensorFlow Research Cloud can be accessed at https://www.tensorflow.org/tfrc.

[30] We follow Google's recommendations for optimal performance and define the size of all arrays as multiples of 128. More information on the architecture of Google's cloud system is available at https://cloud.google.com/tpu/docs/system-architecture.

iteration.

As we describe in Section 2.2, the deterministic age profile in our labor income process is conditional on an initial permanent income percentile $q \in \{25, 50, 75, 90\}$. We solve for optimal behavior separately for each permanent income percentile assuming that both members earn labor income according to the percentile q process.[31] The code that solves for optimal behavior given a permanent income percentile takes approximately 24 hours to run on Google's TensorFlow Research Cloud.

4 Lifecycle portfolio behavior

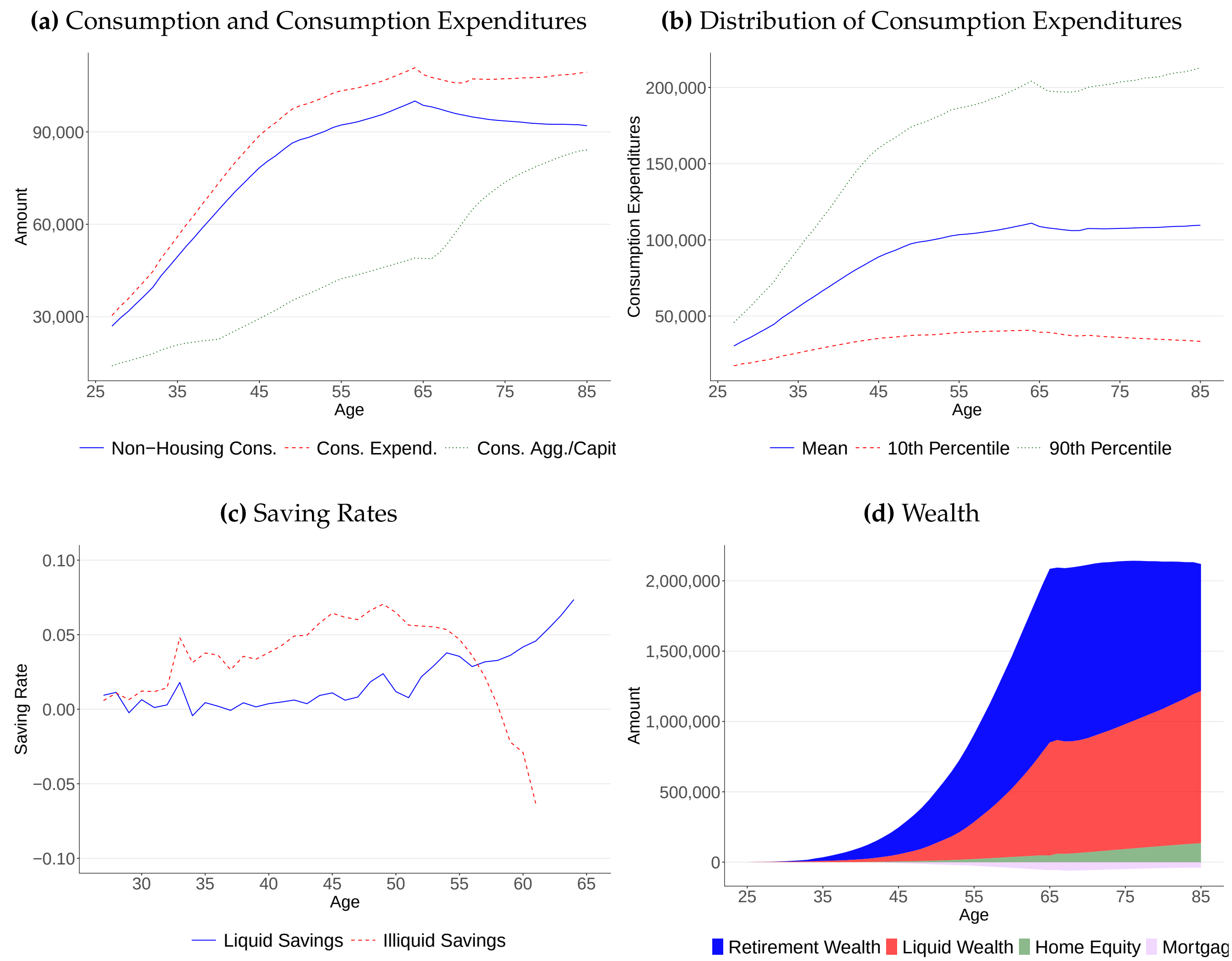
This section characterizes the optimal portfolio choices over the life implied by the model and shows two main results. First, the average share of financial assets optimally invested in equity starts quite low but rises rapidly early in life, peaking around age 45 and declining thereafter. For retirement wealth, the average optimal share invested in equity starts high, and overall looks a lot like the glide path of a typical TDF. Second, however optimal portfolios differ significantly depending on other state variables, most importantly wealth level and the 'aggregate' variables that affect expected stock and bond returns. Section 6 repeats our analysis for impatient households that accumulate less wealth.

As just described in Section 3, our algorithm involves simulating the shocks, state variables, and choices of a large number of households. We use the final simulations at the optimal parameter values (optimal policy functions) to characterize optimal behavior and outcomes. We approximate the population of interest by combining the lifecycles of state variables and actions of households from each of the four percentiles (q) of average income profiles ($\bar{y}^i$). We take a random subset of households from the $q = 25$ group to represent the lowest 37.5% of the permanent income distribution, and appropriately lower number of households for each other percentile so that the $q = 50$ represents the next 25% of the permanent income distribution (between the 37.5th and 62.5th percentiles), the $q = 75$ group represents the 20% of the distribution between the 62.5th and 82.5th percentiles, and the $q = 90$ group represents the top 17.5% of earners.

To interpret dollar amounts, we note again that our model applies to a married, two-earner, couple born in 1924 and amounts are 2013 dollars. We plot choices and outcomes from age 27 (so that initial conditions to not drive some figures) to the end of working life

[31] This amounts to an assumption that assortative marriage matching causes all individuals to have the same permanent earnings capacity as their spouse.

Figure II: Average Consumption, Saving Rates, and Wealth by Age



Notes: Consumption expenditures are defined as: non-housing consumption, health expenditures, house and mortgage transaction costs, and rent or rental equivalence (ϕh). The consumption aggregate per capita is the argument of the CRRA period utility function. Saving rates are averages of x^L/Y and x^I/Y dropping family labor incomes below \$10,000.

or until 85 (a household can survive until age 102 with very low probability).

4.1 Optimal Behavior

Figure II.a plots the average level of non-housing consumption (c which also excludes medical cost shocks), total consumption expenditures (c plus mortgage and home-buying transaction costs, medical expenses, and rent or rental equivalent (ϕh)), and the consumption aggregate per effective family member, $c_t^{\alpha}(\tilde{u}^{o_t(g_t^1+g_t^2)}\phi h_t)/w_t$.[32] Because returns are high relative to households impatience, because of precautionary saving, and because of matching of retirement saving, the first two measures rise rapidly until the mid-40's and all three measures rise until retirement. Non-housing consumption, c, falls after retirement while average total consumption expenditures is roughly flat and the difference between the two series is occasional large medical expenses (which become more likely as people become older). Average per capita consumption rises steadily but more slowly throughout working life and then accelerates in old age largely in response to death of one spouse. Non-housing consumption, c, rises rapidly from an average of just under $\$30,000$ to around $\$85,000$ at age 45, rises more slowly thereafter to peak at about $\$100,000$ at age 65, and then declines during retirement. Total expenditures follows a similar pattern but does not decline in retirement due to medical cost shocks. Again these figure pertain to people retiring in 1989 and are in 2013 dollars.

Figure II.b shows that cross-household dispersion in consumption expenditures increases until around age 45. In mid-life the 10th percentile of expenditures is around $45,000 in a year while the 90th percentile is nearly $200,000. After age 45, consumption inequality rises only slowly and primarily reflecting increasing expenditures at the 90th percentile of the distribution. During retirement, inequality grows due to medical expenses (as well as mortality).

Turning to saving rates, Figure II.c shows that during most of their working lives, households mainly save in retirement accounts. During their early years, it is optimal to save primarily in housing and retirement accounts, with retirement saving rates rising from 1% before age 30 to peak over 5% at age 50. Liquid saving rates rise after age 50, when retirement saving rates decline and go negative before retirement.[33] Figure II.d shows the resulting average wealth accumulated at each age. On average, households build liquid and illiquid wealth steadily and rapidly during their working lives until age 65, after which they on average consume their returns and pension incomes. At 65, average wealth is roughly $2.25 million, with more than half of that in retirement accounts.[34]

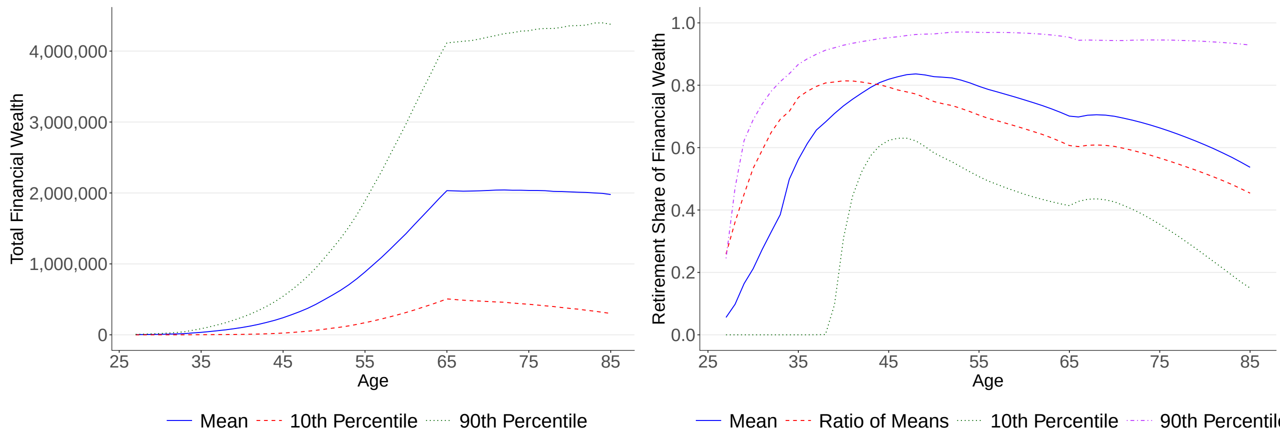
[32]We exclude maintenance costs from consumption expenditures since they are part of the rental equivalence. When a household hits the consumption floor, their consumption expenditures are $\underline{c}$

[33]The average saving rate in retirement accounts turns negative for three reasons: i) some people with high wealth smooth (nonlinear) taxes by starting to withdraw retirement saving before retirement; ii) the retirement contributions of high income households are capped so decrease with income; and iii) high-wealth households who realize low labor income start withdrawing substantial amounts.

[34]Not shown, the share of households that are homeowners remains very low until around age 35, at which

Figure III: Financial Wealth by Age

(a) Distribution of Financial Wealth **(b)** Retirement Share of Financial Wealth



Notes: Financial wealth is liquid wealth plus retirement wealth.

More importantly, underlying these averages are substantial differences in financial wealth across households at every age: during retirement, the 90th percentile of the wealth distribution is double the average, above $4 million, and the 10th percentile is one quarter of the average, only around $500,000 (Figure III.a). Further, wealth inequality rises rapidly until retirement, but continues to grow some as the wealthy earn high returns and save for bequests, while the medical costs matter more for relatively low-wealth households who deccumulate wealth as they age.

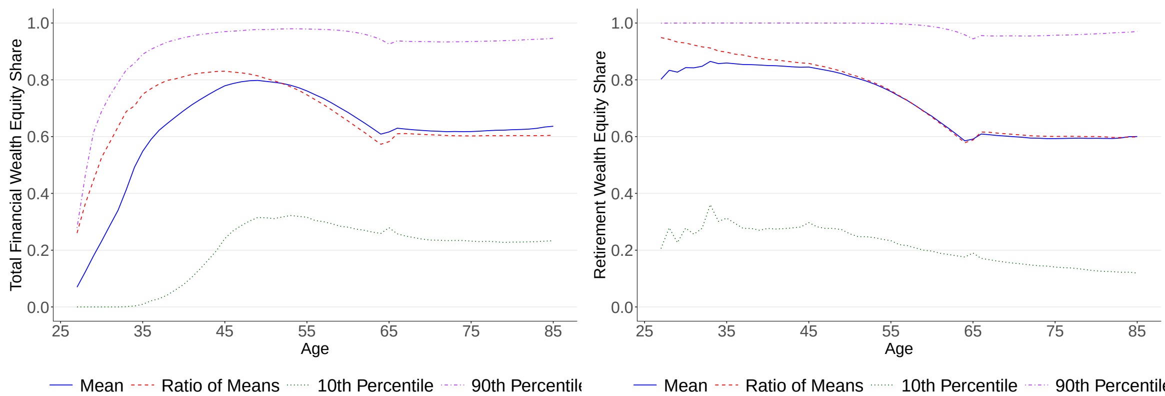
There is also substantial heterogeneity across households in the fraction of their financial wealth held in retirement accounts. Figure III.b shows that young households first build liquidity, and then accumulate retirement wealth so that after age 35, most households hold most of their financial wealth in retirement accounts. However, the top 10 percent of households hold nearly all their wealth in retirement accounts at all ages above 35, while the bottom ten percent at ages below 40 and above 55 have less than half their financial wealth in retirement accounts.

Turning to our main focus, portfolios, the average optimal equity share in total financial wealth is hump-shaped over the lifecycle but the average optimal equity share in retirement accounts echoes the pattern of typical advice and delivered by Target Date Funds. Figure IV.a shows that the share of financial wealth optimally invested in the stock market

age the home ownership rate rises over the next 30 years so that by retirement just over 65% of households own homes. The median and average ratio of mortgage to home value steadily declines from an average of 60% at age 45 to roughly 30% at age 85.

Figure IV: Distribution of Equity Shares of Wealth

(a) Equity Share of Financial Wealth **(b)** Equity Share of Retirement Wealth



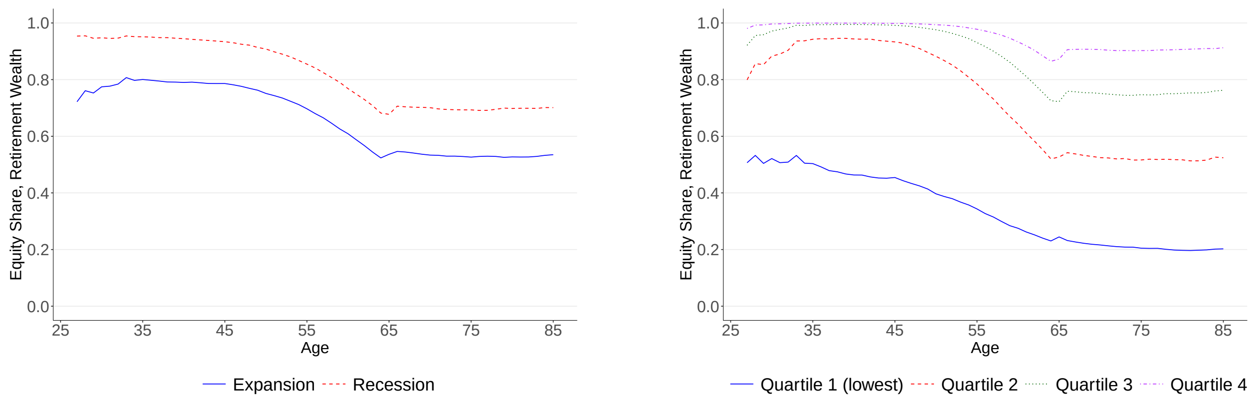
at the beginning of life is below 30%. This low level is driven is entirely by non-retirement wealth, and due significantly to the need for liquidity. The average equity share rises rapidly with age to 80% by the mid-40's, then declines to roughly 60% at retirement and remains steady thereafter.

In retirement accounts, the average optimal share of stock market investment starts high – above 80% – rather than low, and declines steadily with age during the working life, reaching 60% at retirement where is remains during retirement. This declining pattern is similar to that prescribed by TDFs, but the decline in stock holding is less pronounced. Current TDFs typically hold roughly 90 percent of their assets in stocks until roughly age 40 (or 25 years before retirement), at which point the share in stocks declines smoothly to roughly 40 percent at age 75 (ten years after the retirement target date). Figure IV.b shows that the average optimal share declines by less and not at all after retirement.

However, as there was for other household financial choices, there is a lot of variation in the optimal equity share across households at each age, variation that is far from the homogeneous allocations provided by current TDFs. Figure IV.a shows that the 90th percentile of the optimal equity share in financial wealth rises rapidly early in life to reach over 95% in equity at the start of the households' 40's, and remains between 90% and 100% for the remainder of life. For retirement wealth, the 90th percentile of the optimal equity share remains just below 100% until just prior to retirement when it declines slightly to 95%. The 10th percentiles are also far from the average optimal equity share. For retirement wealth, the optimal share invested in equity is around 40% early in life and declines steadily

Figure V: Optimal Equity Shares of Retirement Wealth Over the Business Cycle

(a) Equity Share by State of the Cycle **(b)** Equity Share by Dividend-Price Ratio



with age, falling below 20% at retirement.

These large differences in optimal portfolio shares arise from differences in the economic circumstances. We now turn to the reasons why optimal equity shares vary across people.

4.2 Optimal Equity Shares of Retirement Wealth by State

This subsection shows how the substantial variation in optimal equity share at each age shown in Figure IV relates to differences in economic circumstances. We focus on equity shares in retirement accounts, but Appendix Figure A.2 shows that the differences in equity shares across values of state variables for financial wealth are similar to those for retirement wealth shown in this subsection, just with average lifecycle profiles reflecting Figure IV.a instead of Figure IV.b.

Our model implies that the optimal portfolios of most households, but particularly elderly households are quite sensitive to time-variation in expected returns. Figure V.a shows that equity shares are about 10% higher in a recession than an expansion, and this difference is roughly constant across the age distribution. However, differences in response to variation in the dividend price ratio (correlated with the state of the economy) are larger and differ significantly by age. Figure V.b shows that when the dividend price ratio is in the bottom quarter of its distribution – so expected stock returns are low – the share of retirement wealth invested in equity should be below 40% on average for households over 45 and as low as 20% for households over 70. When the dividend-price ratio is in the top quarter of its distribution, the equity share should be 100% for these households. The

Figure VI: Optimal Equity Shares of Retirement Wealth

optimal equity share of households between age 35 and 45 is relatively similar 75% of the time, only dropping substantially when the dividend price ratio is in its bottom quartile. However, as households age and enter retirement this ceases to be true. Portfolio shares become quite sensitive to expected returns across the distribution of dividend price ratios.

Figure VI.a shows smaller differences in portfolio allocations associated with differences in the level of total financial wealth, with wealthier households having a smoother and more steeply declining lifecycle pattern of equity share. It is optimal for high wealth households to hold more of their portfolios in equity early and life and a lower fraction later in life (these patterns are the same for the equity share of total wealth (Appendix Figure A.2)).

Figure VI.b shows that in mid-life, homeowners in the top quartile of housing leverage hold about 10-15% more of their retirement portfolios in equity than households in the bottom quartile from age 45 to 55, a difference that decays as the differences in leverage wane (and differences in other correlated state variables like wealth start to matter more).

In contrast, there are also only small differences in optimal equity share across percentiles of the income distribution, that is, across different lifecycle profiles of average income (Figure VI.c). Similarly, there are only small variations in the optimal portfolio of equity when households receive good or bad transitory income shocks (Figure VI.d). Finally, as shown in the appendix, optimal equity shares of retirement wealth and of financial wealth do not vary significantly by health or mortality status during retirement (see Appendix Figure A.3).[35]

5 The Welfare Costs and Benefits of Simple Portfolio Rules

Suppose that the household were either not sophisticated enough to make reasonable portfolio choices or not interested in spending the time and effort to make them. How much would the household lose by instead following simply portfolio rules such as maintaining a constant share of assets in equity or following a rule that depends only on age, such as those embedded in current Target Date Funds (TDFs) which now serve as default investment options in most employer-sponsored defined-contribution retirement plans? Or put differently, how much would be gained by moving to more sophisticated portfolio advice?

We begin by considering how much worse the household would do if their retirement saving were allocated to stocks and bonds following the mix prescribed by current a popular current TDF. Specifically, we follow closely the design of the Vanguard Target Retirement Fund series, and impose that up to age 40 retirement wealth is invested 90% in stocks and 10% in bonds. From age 40, the equity share declines 1.5% per year to reach 60% at age 60, then declines by 2.5% per year to reach 49.5% at age 65, and then declines by 3% per year to reach 32% at age 70. The equity share remains at 30% from age 71 onward.[36]

Panel A of Table II shows how much a household would hypothetically pay in percent of consumption (in every state at every age) to be able to (costlessly) follow the optimal

[35] Also, as households transition from renting to home ownership, renters should consistently hold very close to 100% of their retirement wealth in equity while the optimal equity share for homeowners declines linearly from 93% to 78% over these ages on average.

[36] TDFs do not all follow the same glide path, and do not all deliver the same returns, so that different TDFs with the same target date give different returns (Balduzzi and Reuter, 2019) and can have quite different post-fee performance (Shoven and Walton, 2020; Brown and Davies, 2020).

Table II: Consumption certainty equivalent loss of imposing simple rules on retirement portfolio

	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
	Age 25 certainty equivalent					Avg. flow utility certainty equivalent				
Other behaviors		Income percentile					Income percentile			
not re-optimized	All hhs	25th	50th	75th	90th	All hhs	25th	50th	75th	90th
					Panel A: Impose TDF on retirement portfolio					
None	0.45	0.33	0.51	0.52	0.54	1.70	1.58	1.80	1.78	1.73
Liq. portfolio	0.49	0.38	0.52	0.57	0.57	2.01	1.72	2.06	2.25	2.28
All	0.59	0.52	0.63	0.64	0.63	2.82	2.73	2.77	2.90	2.98
					Panel B: Impose constant 2/3 equity share on retirement portfolio					
None	0.49	0.34	0.60	0.60	0.54	1.85	1.58	2.09	2.10	1.80
Liq. portfolio	0.52	0.35	0.60	0.63	0.63	1.85	1.23	2.21	2.23	2.24
All	0.59	0.52	0.63	0.65	0.63	2.23	2.04	2.27	2.40	2.38

Notes: Average family is a weighted average of the given percentiles as described in the text (the 25th, 50th, 75th, and 90th percentile represent, respectively, 37.5%, 25%, 20%, and 17.5% of the population). The certainty equivalent at age 25 is the percent reduction in consumption at all ages and possible outcomes in the original problem that delivers the same expected present discounted value of utility at age 25. The certainty equivalent for average flow utility is the same calculation with $\beta = 1$.

portfolio rules rather than the TDF rules in retirement wealth. Columns 1 – 5 show these figures from the perspective of the household at age 25, first for the average household and then for households at different points in the distribution of deterministic income profiles. This perspective discounts future flow utility at the discount rate $\beta = 0.96$ per year and so puts more weight on flow utility when young than when old. Columns 6 – 10 report the results of the same calculations but for an equally-weighted average of flow utility across the household's life, as if $\beta = 1$, so that the flow utility of the household when old receives equal weight to the flow utility of the household when young.

The utility loss from following a TDF's prescriptions rather than the optimal state-dependent portfolio rule would cost the average household in expectation the equivalent of losing roughly half a percent of all their consumption from the perspective of age 25, and between 1.7 and 2.8 percent of consumption in any year chosen at random. We reach this conclusion from several different calculations.

First, we impose the TDF portfolio on retirement wealth and allow the household to re-optimize all other behaviors. The first row of Panel A shows that the average household loses the equivalent of only 0.45 percent of consumption from the perspective of age 25. But this under-weights the utility of the household when they are old which is when the effects of an incorrect portfolio choice are the largest. That is, from a welfare perspective, this measures the cost to a 25 year old rather than considering equally the perspectives of the family at other ages. Column 6 shows that when weighting flow utility at different ages equally, the consumption equivalent is a much larger 1.7 percent of consumption at all ages and in all state of the world.

However, this first experiment is in some ways inconsistent. We are assuming that the household is somehow unable or unwilling to optimize its retirement wealth portfolio, but at the same time we allow the household to optimize its portfolio choice for non-retirement wealth conditional on holding a TDF in its retirement account. This assumption may understate the true welfare losses because the household in the model adjusts its portfolio in its liquid accounts to 'unwind' the portfolio imposed by the TDF on their retirement account, while households in reality may not. Thus we experiment with not allowing the household to re-optimize their portfolio behavior in liquid accounts.

Row 2 of Table II shows the welfare costs relative to the unconstrained optimum, of imposing a TDF on retirement account, having the household allocate its liquid wealth among assets as it would in the fully-optimal solution, and allowing it to re-optimize all other behavior given these portfolio rules. The costs of imposing the TDF portfolio on retirement wealth under these assumptions are indeed larger. However, the welfare loss from this additional constraint is a trivial 2 basis points of consumption from the perspective of age 25, and is just over 0.3 percent using the equally-weighted lifetime perspective. The latter reduction is significant, but the majority of the welfare gains from moving to the fully-optimal portfolio remain. There are two reasons for this finding. First, most households accumulate the majority of their financial wealth in their retirement accounts. Second, the cash in advance constraint keeps low-wealth households from taking a large amount of risk.

An alternative way to impose that the household does not unwind the TDF allocations using liquid wealth is to impose that the household simply uses the optimal unconstrained decision rules for all choices other than its portfolio of retirement wealth. That is, we impose the TDF shares in retirement wealth and keep the remaining behaviors of the household for all other decision variables – optimized for the fully-optimal portfolio in the retirement portfolio – the same (conditional on state variables). As shown in the third row of Panel A of Table II, this leads to a loss equivalent to roughly 0.59% of consumption from the perspective of household at age 25 and to a larger 2.82% of consumption for the average household simply averaged across all years of life.

Panel A of Table II shows that in all three of these measures, the potential welfare gains from moving to optimal portfolio behavior in the retirement account tend to be larger for higher-income households, who are those in the upper middle class. Because high net worth individuals tend to use personal financial advisers and because low income households have less access to 401k plans (or are offered plans with lower match rates than we have modelled), this is a segment of the population that has leaned heavily on TDFs.

Finally, how do the benefits of TDFs compare to those of simpler, non-age dependent rules that were commonly used prior to the rise of age-dependent advice embedded in TDFs? Panel B of Table II shows that the glide path of the TDF does not actually improve outcomes for the average investor relative to the common, courser advice to maintain a constant two-thirds share of wealth in equity, advice still embedded in some balanced funds. Panel B of Table II shows that the consumption-based welfare losses of a constant equity share are generally lower than those associated with using a TDF. This somewhat surprising result stems from the fact that TDFs tend to decrease equity shares by too much as people age, so that the benefits of a constant equity share of 2/3 for older households is significant, and outweighs the benefit of the TDF guidance to hold higher equity shares early life.

To summarize our findings, while TDFs may lead household to avoid worse mistakes, because they impose the same portfolio on everyone of the same age, there is scope for substantial improvement – 2 to 3 percent of consumption – from more individualized financial advice or from more customized TDFs.

6 Portfolio Choice with Lower Wealth Accumulation

The previous section evaluates prescriptive portfolio advice and the benefits to further customization in a model in which all household decisions besides portfolio choices are optimal. However, it is reasonable to consider whether households that cannot choose optimal portfolios might make other sub-optimal decisions. In particular there is substantial concern that many households do not save sufficient wealth for retirement. While households in our model consist of two-earner, stable families, they accumulate quite a bit more wealth than the average household in the cohort that we observe one particular draw for in reality. Households in our simulation accumulate on average about $2 million in total wealth by age 65 (Figure II.d), while, measured across all types of households, the same cohort in reality accumulated mean net worth of around $600,000 in 2013 dollars according to the 1989 and 1992 waves of the Survey of Consumer Finances.

In this section, we analyze an alternative model in which households make decisions as if they were more impatient and have a discount rate of $\beta = 0.93$, rather than $\beta = 0.96$. While our original specification of time preferences is consistent with the dynamic models that we build upon (e.g. DFJ estimate and use $\beta = 0.97$), there are many other lifecycle models that deliver less saving by many or some households due for example to present bias (Laibson, 1997; Laibson, Repetto, and Tobacman, 2007) or some more impatient households (Samwick, 1998; Warner and Pleeter, 2001; Hendricks, 2007).

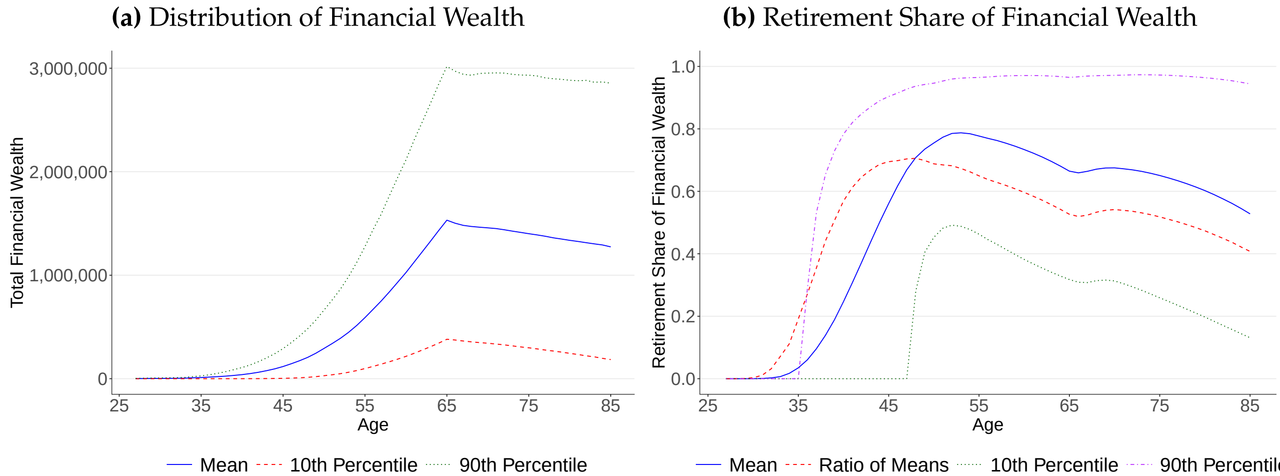
In this alternative specification, TDFs continue to perform relatively well but with some notable differences from the previous section. First, while optimal equity shares of retirement wealth remain similar when agents have accumulated substantial wealth, they actually rise with age prior to age 45. Second, optimal equity shares respond to the same state variables but by more, which implies that, third, the welfare losses of simple TDF rules relative to optimal behavior are somewhat larger. Thus, when households save less and accumulate less wealth, there are larger benefits to further customization of portfolio shares across broad asset classes.

6.1 Lifecycle Behavior

Figure VII shows that a reduction in the discount factor to $\beta = 0.93$ delays the age at which households begin to save into their illiquid retirement accounts and substantially reduces lifecycle wealth accumulation. Whereas the mean household in the baseline model has about \$2 million in total financial wealth at the retirement age of 65, the corresponding mean wealth accumulation in Figure VII.a is 25% lower at about \$1.5 million. Most of this reduction comes from eliminating a right tail of very high wealth accumulation: while the 90th-percentile line in Figure III.a is above \$4 million after age 65, the 90th percentile in Figure VII.a never exceeds \$3 million during the retirement period. The delay in retirement saving is even more apparent. While 90% of households in the baseline model have nonzero retirement savings by age 27, the 90th-percentile line in Figure VII.b remains at zero until age 35. The bottom 10% of households do not contribute to their retirement accounts until age 48.

The equity-share patterns in Figure VII.a show that, from age 45 on, the average optimal share of retirement wealth invested in the stock market is quantitatively similar to that in the baseline model: declining from just below 90% to 60% during retirement. In the

Figure VII: Financial Wealth by Age, with Lower Wealth Accumulation



Notes: Financial wealth is liquid wealth plus retirement wealth.

baseline model, from age 25 to 45, equity shares are always flat or decreasing. However, when households are more impatient, less than 10% of households accumulate more than $1,000 in retirement wealth before age 35, and from age 35 to 45 the average optimal equity share starts at 75% and grows to 87% as more households build more retirement wealth. This increase is primarily driven by households with little retirement wealth (green dotted line in Figure VIII.a).

Figures VIII.b and VIII.c demonstrate that, as was the case for more patient households, optimal equity shares are higher in recessions and are increasing in the dividend-price ratio, but the magnitude of these differences are greater than was the case for more patient households. The most noticeable change that accompanies less wealth accumulation is the difference in optimal portfolios with wealth. The qualitative pattern in Figure VIII.d is unchanged (with the top half of the wealth distribution having higher equity shares before age 45 and lower equity shares thereafter), but wealth heterogeneity is more important quantitatively: the top quartile invests 23 percentage points more in equity at age 36 than the bottom quartile, and 26 percentage points less in equity at age 65. This result follows from the fact that more concave utility functions tend to accentuate differences in optimal behavior across the wealth distribution when average wealth and consumption levels are lower.

Figure VIII: Optimal Equity Shares of Retirement Wealth, with Lower Wealth Accumulation

Notes: Data series are only plotted for ages where at least 10% of households have at least \$1,000 in retirement wealth.

6.2 Welfare Implications of Simple Portfolio Rules

When households accumulate less wealth, the welfare costs of the imperfect portfolio rules embedded in TDF-type strategies are larger relative to fully-optimal behavior. And unlike in the previous section, these costs are larger for lower wealth households and are larger than a simpler age-independent 2/3 equity rule.

Two opposing forces govern the relative magnitudes of the consumption-equivalent welfare losses in the baseline versus the low-wealth model. First, since the discount factor β is smaller in the low-wealth model, portfolio-allocation decisions that affect later-in-life consumption levels have a mechanically smaller effect on lifetime utility from the perspective of a discounting household at age 25. The welfare losses shown in columns

Table III: Consumption certainty equivalent loss of imposing simple rules on retirement portfolio, with lower wealth accumulation

	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
	Age 25 certainty equivalent					Avg. flow utility certainty equivalent				
Other behaviors		Income percentile					Income percentile			
not re-optimized	All hhs	25th	50th	75th	90th	All hhs	25th	50th	75th	90th
	Panel A: Impose TDF on retirement portfolio									
None	0.15	0.14	0.15	0.20	0.13	2.25	2.58	2.56	2.04	1.34
Liq. portfolio	0.19	0.18	0.23	0.18	0.15	3.32	3.61	3.53	3.49	2.20
All	0.20	0.19	0.23	0.20	0.15	3.33	3.61	3.59	3.49	2.20
	Panel B: Impose constant 2/3 equity share on retirement portfolio									
None	0.07	0.05	0.06	0.06	0.11	1.03	1.84	0.47	0.16	1.08
Liq. portfolio	0.15	0.14	0.17	0.18	0.12	2.12	2.30	2.43	1.99	1.43
All	0.17	0.17	0.20	0.20	0.12	2.22	2.33	2.41	2.47	1.43

Notes: Average family is a weighted average of the given percentiles as described in the text (the 25th, 50th, 75th, and 90th percentile represent, respectively, 37.5%, 25%, 20%, and 17.5% of the population). The certainty equivalent at age 25 is the percent reduction in consumption at all ages and possible outcomes in the original problem that delivers the same expected present discounted value of utility at age 25 (with the discount factor set at $\beta = 0.93$). The certainty equivalent for average flow utility is the same calculation with $\beta = 1$ (and behaviors still re-optimized using $\beta = 0.93$).

1-5 of Table III are therefore generally smaller than the corresponding figures in Table II. However, when average wealth and consumption levels are lower, portfolio choice is more consequential and the constraints imposed by the simple allocation rules have higher flow utility costs in any given time period. The losses to average flow utility in columns 6-10 (where all time periods are treated equally) are thus larger than in the baseline model: while the certainty-equivalent loss of the TDF portfolio for all households is 1.70-2.82% in the baseline model, the corresponding magnitudes in Table III are noticeably larger at 2.25-3.33%.

Interestingly, the constant 2/3-equity portfolio now outperforms the TDF portfolio, with the consumption-equivalent losses in Panel B of Table III systematically smaller than those in Panel A. This result stems from the fact that optimal equity shares early in life are smaller with lower wealth accumulation (see again Figure VIII), moving average optimal behavior further from the more aggressive TDF portfolio and closer to the constant 2/3-equity rule. During retirement, the optimal portfolio allocation is slightly lower, so not as far from the TDF allocation. But the optimal equity share remains closer to the 2/3 equity rule and this effect is not strong enough to overcome the former effect.

In sum, when households accumulate less wealth, there are larger gains to further individual customization of portfolios across broad asset classes, particularly based on individual investor wealth.

7 Concluding Remarks

This paper illustrates how recent advances in the algorithms, software, and hardware of machine learning can be used to vastly increase the complexity of problems that economists can solve quantitatively, and therefore enable the field to provide more rigorous quantitative evaluation of the simple rules derived from quantification of parsimonious models. We also illustrate how such an approach can be used to investigate improvements in simple rules by solving for constrained-optimal policy functions over a subset of states.

We apply this idea to a lifecycle problem and study household portfolio choice when the model environment includes many features of reality studied only independently – one or two at a time – in previous work. We find that the simple rules embedded in TDFs approximate reasonably well the average age-specific optimal portfolio rules or our model, although for the population that we study, TDFs impose lower investment in the stock market at older ages than our model recommends.

We see several areas for future work. First, one might enrich the model to include more heterogeneity in labor income processes, different exposures of labor income to aggregate returns, or long-run correlation between dividends and labor income. Second, it would be interesting to also consider a wider choice in financial instruments, such as the choice between traditional and Roth IRA/401k's or the option of various types of annuities or life insurance (as in Yogo, 2016; Koijen et al., 2016). Finally, it would be interesting to see how robust our results are to other realistic assumptions about utility that have been studied in lifecycle models of portfolio choice, such as habits (Gomes and Michaelides, 2003), luxury goods (Wachter and Yogo, 2010), hyperbolic discounting (Angeletos, Laibson, Repetto, Tobacman, and Weinberg, 2001; Love and Phelan, 2015), flow utility from information (Pagel, 2018), risk aversion that declines with age or wealth (Meeuwis, 2019), or incorporating flow disutility from anxiety about future uncertainty as in Epstein-Zin preferences.

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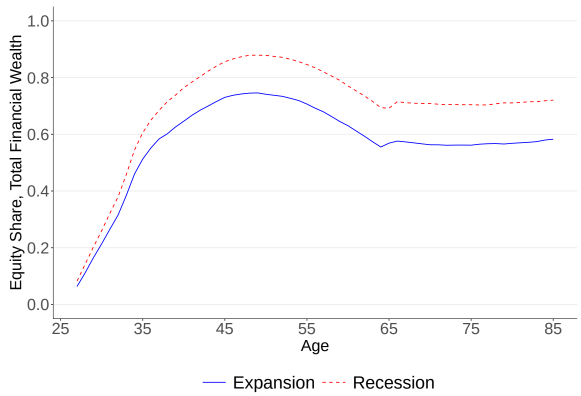
Appendix

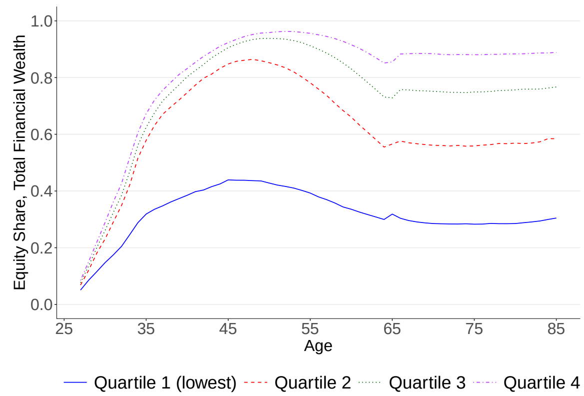
A Additional Figures

Figure A.1: Optimal Equity Shares of Financial Wealth

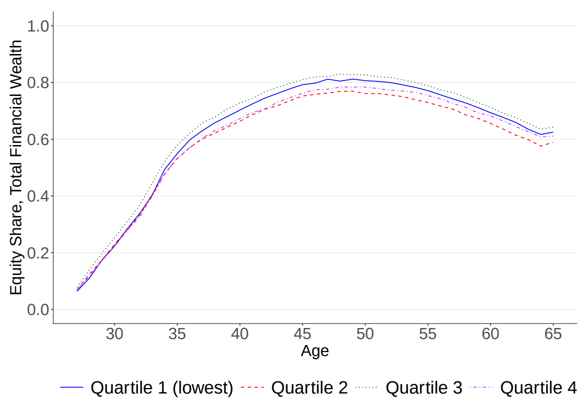
(a) Equity Share by State of the Cycle



(b) Equity Share by Dividend-Price Ratio



(c) Equity Share by Transitory Income



(d) Equity Share by Permanent Income

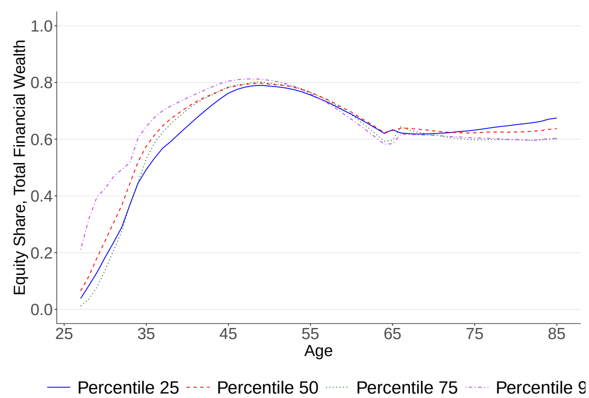
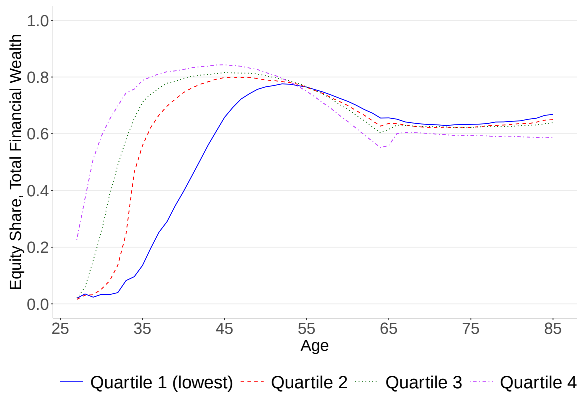


Figure A.2: Optimal Equity Shares of Financial Wealth

(a) Equity Share by Total Financial Wealth



(b) Equity Share by Housing Leverage

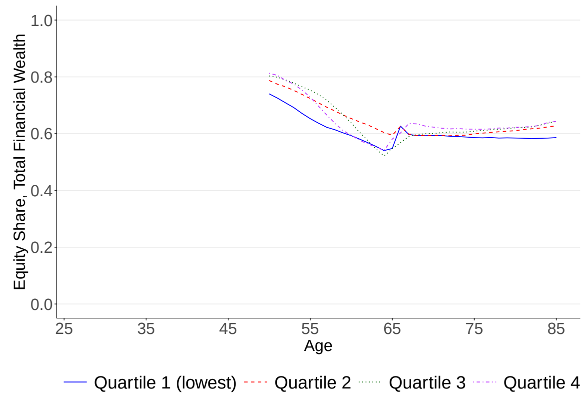
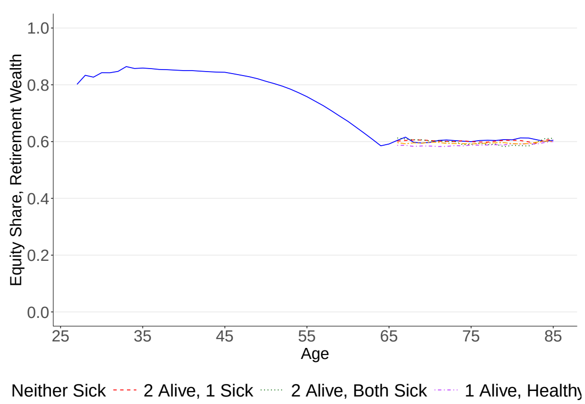


Figure A.3: Optimal Equity Shares of Wealth by Health, Mortality

(a) By Health and Mortality; Retirement Wealth



(b) By Health and Mortality; Financial Wealth